Zeros of Polynomial Functions

Theorem 1: **The Fundamental Theorem of Algebra**

If $f(x)$ is a polynomial of a degree "n", where $n>0$, then "f" has at least one zero in the complex number system.

Using the fundamental theorem of Algebra and the equivalence of zeros and factors, you obtain the Linear Factorization Theorem.

Theorem 2: **Linear Factorization Theorem**

If $f(x)$ is a polynomial function of degree "n" where $n>0$, then "f" has precisely "n" linear factors.

$f(x)=a_n(x-c_1)(x-c_2)(x-c_3)...(x-c_n)$; where $c_1,c_2,c_3,c_4,...c_n$ are complex numbers.

In particular if the leading coefficient of the polynomial function is 1 then

$f(x)=(x-c_1)(x-c_2)(x-c_3)...(x-c_n)$

The zeros of $f(x)$ are $c_1,c_2,c_3,c_4,...c_n$.

The repetitions of a zero c_i is based on the multiplicity of c_i .

a_n is the leading coefficient of $f(x)$.

Example 1: Find the polynomial function of degree 3 having roots 1,-2and 4, such that $f(-1)=3$

The polynomial function is of degree 3, so according to the linear factorization theorem

$f(x)=a(x-c_1)(x-c_2)(x-c_3)$

The zeroes are 1, -2, and 4

$\Rightarrow c_1=1; c_2=4 \text{ and } c_3=-2$

$f(x)=a(x-1)(x-4)(x+2)$

We still have one unknown which is a, $f(-1)=3$

$\Rightarrow f(-1)=a(-1-1)(-1-4)(-1+2)$

$\Rightarrow 3=a(-2)(-5)(1)$

$\Rightarrow 3=10a$

$\Rightarrow a=\frac{3}{10}$

Therefore, $f(x)=\frac{3}{10}(x-1)(x-4)(x+2)$

Expanding $f(x)$, we will obtain $f(x)=\frac{3}{10}x^3-\frac{9}{10}x^2-\frac{9}{5}x+\frac{12}{5}$

Example 2: Find a polynomial function of degree 3, having the zeros 1, $3i$, and $-3i$.

$f(x)=a\ (x-1)(x-3i)(x+3i)$.

The number a can be any nonzero number. The simplest function will be obtained if we let $a=1$

Then we have

$$\begin{aligned} f(x) &= (x-1)(x-3i)(x+3i) \\ &= (x-1)(x^2+9) \\ &= x^3+9x-x^2-9 \\ &= x^3-x^2+9x-9 \end{aligned}$$

The Fundamental Theorem of Algebra and the Linear Factorization Theorem tell you only that the zeros or factors of a polynomial exist, not how to find them. Such theorems are called existence theorems. To find the zeros of polynomial function, we must rely on other techniques.

The Rational Zero Test relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

Keep in mind: a rational number is of the form $\frac{a}{b}$, *where* $b \neq 0$

Rule 1: **The Rational Zero Test: If the polynomial** $f(x)=a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2 x^2 + ax + a_0$ has integer coefficients, every rational zero of f has the form $\frac{p}{q}$

Rational zero = $\frac{p}{q}$

where "p" and "q" have no common factors other than 1, and

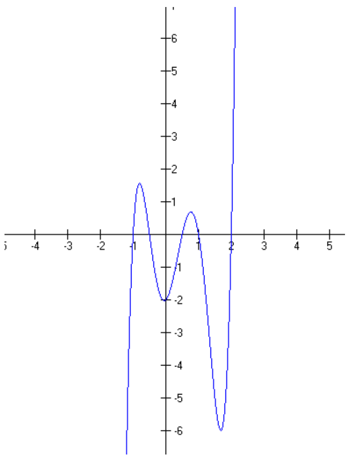
p = a factor of the constant term a_0

q = a factor of the leading coefficient a_n

$$Possible\ rational\ zeros = \frac{Factors\ of\ constant}{Factors\ of\ leading\ coefficient}$$

Example 3: Find the possible rational zeros of the function:

$f(x)=4x^5-8x^4-5x^3+10x^2+x-2$



The constant coefficient is 2 $\Rightarrow$ **the factors of 2 are** $\{1,-1,2,-2\}$

The leading coefficient is 4 $\Rightarrow$ **the factors of 4 are** $\{1,-1,2,-2,4,-4\}$

$\Rightarrow$ The possible rational zeros = $\frac{(1,-1,2,-2)}{(1,-1,2,-2,4,-4)}$ so,

$\Rightarrow$ The possible zeros are: $\left\{1,-1,\frac{1}{2},\frac{-1}{2},\frac{1}{4},\frac{-1}{4},2,-2\right\}$

Looking at the graph of the function, we can see that the real zeros are: $(-1,0),\left(-\frac{1}{2},0\right),\left(\frac{1}{2},0\right),(1,0)\ and\ (2,0)$

Theorem 3: **Conjugate Pairs Theorem**

Let $f(x)$ be a polynomial function that has **real coefficients**. If $a+bi$, where $b \neq 0$, is a zero of the function, then the conjugate $a-bi$ is also a zero of the function.

Example 4: Find all the zeros of $g(x)=x^3+81x$ if $9i$ is a root.

The coefficients of $g(x)$ are 1 *and* 81, that are real coefficient $\Rightarrow$ The conjugate pairs theorem is applicable.

If $9i$ is a root, then $-9i$ is a root.

$\Rightarrow g(x)=x^3+81x=x(x-9i)(x+9i)$

Example 5: **Find the roots of** $f(x)=4x^5-8x^4-5x^3+10x^2+x-2$

The possible zeros are:

$\left\{1,-1,\frac{1}{2},\frac{-1}{2},\frac{1}{4},\frac{-1}{4},2,-2\right\}$

Try $x=-1$,

-1	4	-8	-5	10	1	-2
		-4	12	-7	-3	2
	4	-12	7	3	-2	0

$\Rightarrow f(x)=(x+1)(4x^4-12x^3+7x^2+3x-2)$

Now we will divide $4x^4-12x^3+7x^2+3x-2$ **by** $x-1$

1	4	-12	7	3	-2
		4	-8	-1	2
	4	-8	-1	2	0

So, $f(x)=(x+1)(x-1)(4x^3-8x^2-x+2)$

2	4	-8	-1	2
		8	0	-2
	4	0	-1	0

Therefore, $f(x)=(x+1)(x-1)(x-2)(4x^2-1)$

$f(x)=(x+1)(x-1)(x-2)(2x-1)(2x+1)$

$So, x=-1, x=1, x=2, x=-\frac{1}{2}, x=\frac{1}{2}$