Vectors and Dot Products

When we multiply two vectors together we do not obtain another vector. Instead, we obtain a scalar. This product is referred to as the **dot product**.

Definition 1: Let $u = \langle u_1, u_2 \rangle = u_1\mathrm{i} + u_2\mathrm{j}$ and $v = \langle v_1, v_2 \rangle = v_1\mathrm{i} + v_2\mathrm{j}$. The **dot product** of **u** and **v**, denoted $u \bullet v$, is defined as:

$$u \bullet v = u_1 v_1 + u_2 v_2$$

Example 1: Find each dot product.

a. $\langle 4,5 \rangle \cdot \langle 2,3 \rangle$

$$\langle 4,5 \rangle \cdot \langle 2,3 \rangle = 4(2) + 5(3)$$
$$= 8 + 15$$
$$= 23$$

b. $\langle 2,-1 \rangle \cdot \langle 1,2 \rangle$

$$\langle 2,-1 \rangle \cdot \langle 1,2 \rangle = 2(1) + (-1)(2) = 2 - 2 = 0$$

c. $\langle 0,3 \rangle \langle 4,-2 \rangle$

$$\langle 0,3 \rangle \cdot \langle 4,-2 \rangle = 0(4) + 3(-2) = 0 - 6 = -6$$

Properties: *Properties of the Dot Product*: If **u**, **v**, and **w** are vectors and *m* is a real number then:

1) $u \bullet u = \|u\|^2$
2) $u \bullet v = v \bullet u$
3) $u \bullet (v + w) = u \bullet v + u \bullet w$
4) $(mu) \bullet v = m(\mathrm{u} \bullet v) = u \bullet (mv)$
5) $0 \bullet u = 0$

The angle between two nonzero vectors is the angle θ, $0 \le \theta \le \pi$ between their respective standard position vectors, this angle can be found using the dot product.

Theorem 1: *Alternative form of dot product:* If θ is the angle between two nonzero vectors **a** and **b**, then $a \bullet b = \|a\|\|b\|\cos\theta$.

Theorem 2: *Cosine of the Angle Between Vectors*: If θ is the angle between two nonzero vectors **a** and **b**, then $\cos\theta = \dfrac{a \bullet b}{\|a\|\|b\|}$

Theorem 3: *Orthogonal Vectors*: Two vectors **u** and **v** are orthogonal if and only if $u \bullet v = 0$.

Definition 2: *Parallel and Orthogonal Vectors*: Let θ be the angle between two nonzero vectors **u** and v. Then, by definition:

1) **u** and **v** are **parallel** if $\theta = 0$ or $\theta = \pi$ $\quad (\cos\theta = \pm 1)$
2) **u** and **v** are **orthogonal** if $\theta = \dfrac{\pi}{2}$. $\quad (\cos\theta = 0)$