Using Fundamental Identities

We studied before the basic definitions and some properties of the individual trigonometric functions. In this lesson, we will use the fundamental identities to do the following:

1) Evaluate trigonometric functions
2) Simplify trigonometric expressions
3) Develop additional trigonometric identities

Let us derive the identities and we will start from the basic definitions that we learned earlier.
sin θ and cos θ are ratios defined as:

$\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$

Now, we use these results to find an important definition for tan θ:

$$\tan\theta = \frac{y}{x}$$

$$\tan\theta = \frac{y \div r}{x \div r}$$

$$\Rightarrow \tan\theta = \frac{\frac{y}{r}}{\frac{x}{r}}$$

$$But \sin\theta = \frac{y}{r} ....and..... \cos\theta = \frac{x}{r}$$

$$\Rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Identity 1: $\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$

Using the Pythagorean Theorem we obtain: $r^2 = x^2 + y^2$

Dividing through by r^2 gives us:

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\Rightarrow \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$But \sin\theta = \frac{y}{r} ....and..... \cos\theta = \frac{x}{r}$$

$$\Rightarrow (\sin\theta)^2 + (\cos\theta)^2 = 1$$

$$\therefore \sin^2\theta + \cos^2\theta = 1$$

Identity 2: $\sin^2\theta + \cos^2\theta = 1$

We now proceed to derive two other related formulas that can be used when proving trigonometric identities.

Dividing $\sin^2\theta + \cos^2\theta = 1$ through by $\cos^2\theta$ gives us:

$$\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

$$\Rightarrow \frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

$$\Rightarrow \left(\frac{\sin\theta}{\cos\theta}\right)^2+\frac{\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}}=\frac{1}{\cos^2\theta}$$

$$\Rightarrow (\tan\theta)^2+1=\left(\frac{1}{\cos\theta}\right)^2$$

$$\therefore \tan^2\theta+1=\sec^2\theta$$

Identity 3: $\tan^2\theta+1=\sec^2\theta$

Dividing $\sin^2\theta+\cos^2\theta=1$ through by $\sin^2\theta$ gives us:

$$\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$

$$\Rightarrow \frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$

$$\Rightarrow \frac{\cos^2\theta}{\sin^2\theta}+\frac{\sin^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$

$$\Rightarrow \left(\frac{\cos\theta}{\sin\theta}\right)^2+\frac{\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}}=\frac{1}{\sin^2\theta}$$

$$\Rightarrow (\cot\theta)^2+1=\left(\frac{1}{\sin\theta}\right)^2$$

$$\therefore \cot^2\theta+1=\csc^2\theta$$

Identity 4: $\cot^2\theta+1=\csc^2\theta$

Fundamental Trigonometric Identities:

Reciprocal Identities:

$$\sin\theta=\frac{1}{\csc\theta} \qquad \cos\theta=\frac{1}{\sec\theta} \qquad \tan\theta=\frac{1}{\cot\theta}$$

$$\csc\theta=\frac{1}{\sin\theta} \qquad \sec\theta=\frac{1}{\cos\theta} \qquad \cot\theta=\frac{1}{\tan\theta}$$

Pythagorean Identities:

$$\sin^2\theta+\cos^2\theta=1 \qquad \tan^2\theta+1=\sec^2\theta \qquad \cot^2\theta+1=\csc^2\theta$$

Quotient Identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Example 1: Use the basic trigonometric identities to determine the other five values of the trigonometric functions given that $\sin\alpha = \frac{7}{8}$ and $\cos\alpha < 0$

$\sin^2\alpha + \cos^2\alpha = 1$

$\Rightarrow \left(\frac{7}{8}\right)^2 + \cos^2\alpha = 1$

$\Rightarrow \cos^2\alpha = 1 - \left(\frac{7}{8}\right)^2$

$\Rightarrow \cos^2\alpha = 1 - \frac{49}{64} = \frac{15}{64}$

$\cos\alpha = \pm\sqrt{\frac{15}{64}} \quad \text{but } \cos\alpha < 0$

$\Rightarrow \cos\alpha = -\sqrt{\frac{15}{64}}$

$\sin\alpha = \frac{7}{8} \quad \text{and} \quad \cos\alpha = -\sqrt{\frac{15}{64}} = \frac{-\sqrt{15}}{8}$

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{7/8}{-\sqrt{15}/8} = \frac{-7}{\sqrt{15}} = \frac{-7\sqrt{15}}{15}$

$\cot\alpha = \frac{1}{\tan\alpha} = \frac{1}{-7/\sqrt{15}} = \frac{-\sqrt{15}}{7}$

$\csc\alpha = \frac{1}{\sin\alpha} = \frac{8}{7}$

$\sec\alpha = \frac{1}{\cos\alpha} = \frac{1}{-\sqrt{15}/8} = -\frac{8}{\sqrt{15}} = -\frac{8\sqrt{15}}{15}$