

## Trigonometric Functions of Any Angle

The definition of trigonometric functions was restricted to acute angles. Here, the definition will be extended to cover any angle.

**Definition:** Let  $\theta$  be an angle in standard position with  $(x, y)$  a point on the terminal side of  $\theta$  and

$$r = \sqrt{x^2 + y^2} \neq 0$$

$$\sin \theta = \frac{y}{r}$$

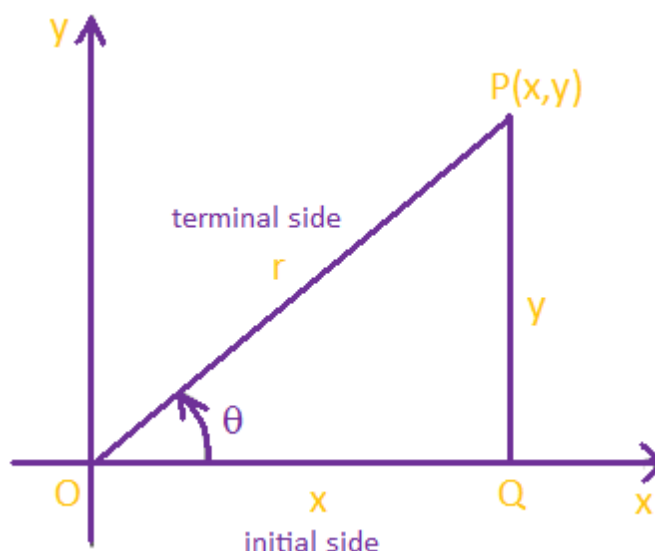
$$\cos \theta = \frac{x}{r}$$

$$\tan \theta = \frac{y}{x}; x \neq 0$$

$$\cot \theta = \frac{x}{y}; y \neq 0$$

$$\sec \theta = \frac{r}{x}; x \neq 0$$

$$\csc \theta = \frac{r}{y}; y \neq 0$$



Because  $r = \sqrt{x^2 + y^2}$  cannot be zero, it follows that the sine and cosine functions are defined for any real value of  $\theta$ . However, if  $x = 0$ , the tangent and secant of  $\theta$  are undefined. For example the tangent of  $90^\circ$  is undefined. Similarly, if  $y = 0$ , the cotangent and cosecant of  $\theta$  are undefined.

**Note:** To find  $r$ , we are actually using the Pythagorean Theorem, since we have a right angled triangle.

**Example 1:** Let  $(-3, 4)$  be a point on the terminal side of  $\theta$ . Find the values of the six trigonometric functions of  $\theta$

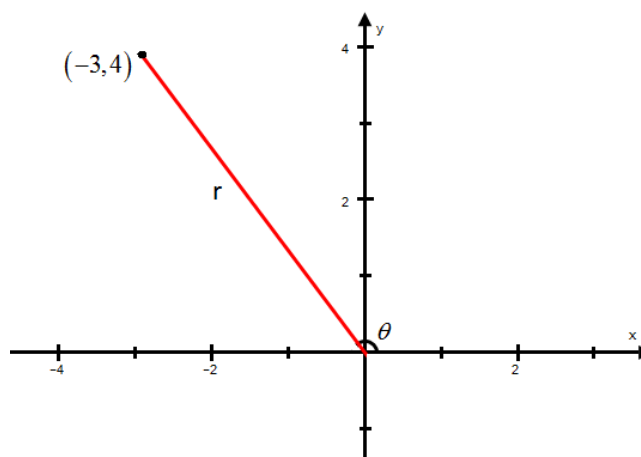
$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2}$$

$$\Rightarrow r = \sqrt{9+16} = \sqrt{25} = 5$$

$$\sin \theta = \frac{y}{r} = \frac{4}{5}; \cos \theta = \frac{x}{r} = -\frac{3}{5}$$

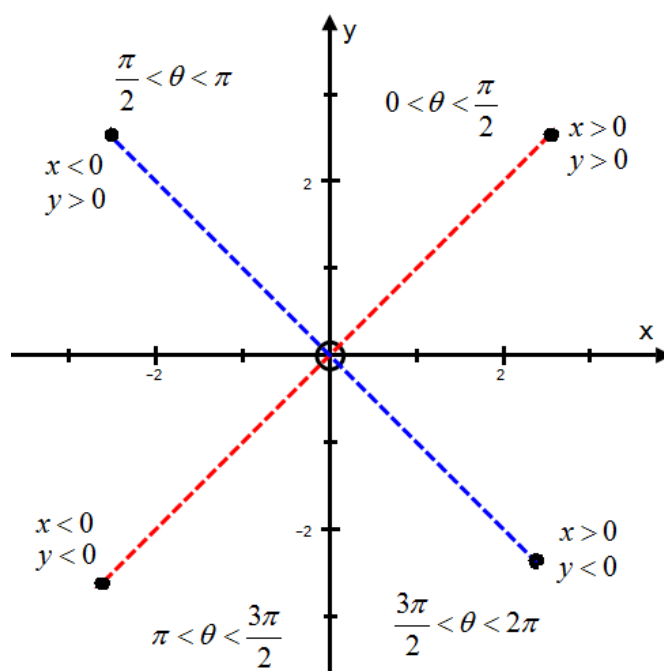
$$\tan \theta = \frac{y}{x} = -\frac{4}{3}; \cot \theta = \frac{x}{y} = -\frac{3}{4}$$

$$\sec \theta = \frac{r}{x} = -\frac{5}{3}; \csc \theta = \frac{r}{y} = \frac{5}{4}$$



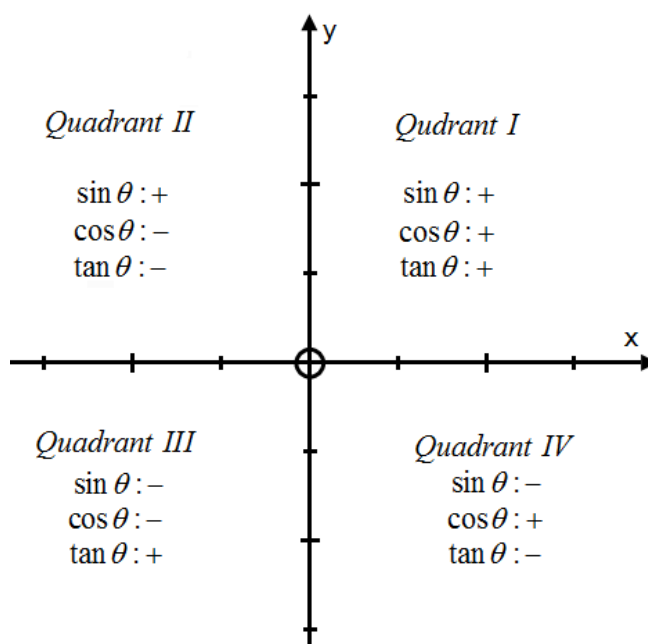
### Signs of Function Values

We can tell the interval of the angle  $\theta$  from the signs of the coordinates of the given point. In the previous example, the point  $(-3,4)$  is in the second quadrant so the value of the angle  $\theta$  is between  $90^\circ$  and  $180^\circ$ . Also, we can deduce the signs of the trigonometric functions. The diagram shows the interval of the angle and the relationship with the signs of the x-values and y-values.



The sign of  $r$  is positive all the time because the square root of any number is positive, so, the sign of the sine function is the same as the sign of the y coordinate of the point and the sign of the cosine function is the same as the sign of the x-coordinate of the point.

Therefore, in quadrants I and IV  $\cos \theta > 0$  and in quadrants II and III  $\cos \theta < 0$ . Similarly, we can deduce the signs of the other trigonometric functions. In the diagram the signs of  $\sin \theta$ ;  $\cos \theta$ ;  $\tan \theta$  are shown in each quadrant.



What is a quick way of finding the signs of trig values for angles in a particular Quadrant?

“ASTC”

Think: “All Students Take Calculus”. Start in Quadrant I and progress counterclockwise through the Quadrants:

S	A
T	C

All of the six basic trig functions are positive in Quadrant I.

(They are all positive for acute angles.)

Sin and its reciprocal, Csc, are positive in Quadrant II.

(The other four functions are negative.)

Tan and its reciprocal, Cot, are positive in Quadrant III.

Cos and its reciprocal, Sec, are positive in Quadrant IV.

### Reference Angles

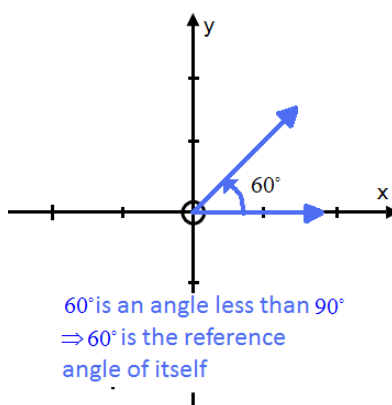
The values of the trigonometric functions of angles greater than  $90^\circ$  or less than  $0^\circ$  can be determined from their values at corresponding acute angles called **reference angles**.

**Definition:** Let  $\theta$  be an angle in standard position. Its reference angle is the acute angle  $\theta'$  formed by the terminal side of  $\theta$  and the horizontal axis.

Assume angle  $A$  is positive and less than  $360^\circ$  ( $2\pi$ ), we have 4 possible cases:

1. If angle  $A$  is in quadrant I then the reference angle.

$$A_r = A$$

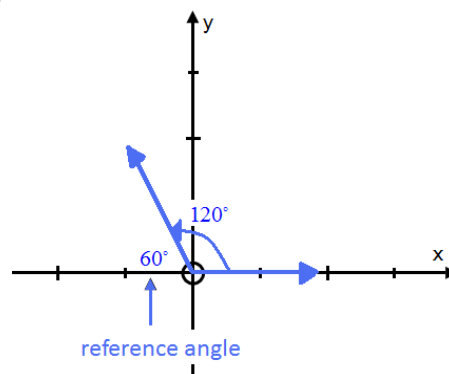


2. If angle  $A$  is in quadrant II then the reference angle

$$A_r = 180^\circ - A \text{ if } A \text{ is given degrees}$$

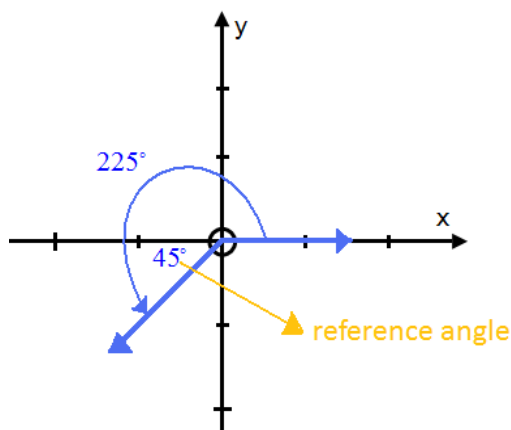
and

$$A_r = \pi - A \text{ if } A \text{ is given in radians.}$$



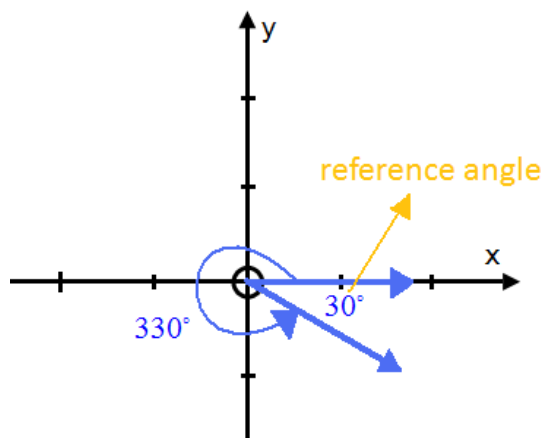
3. If angle A is in quadrant III then the reference angle

$A_r = A - 180^\circ$  if A is given degrees  
and  
 $A_r = A - \pi$  if A is given in radians.



4. If angle A is in quadrant IV then the reference angle

$A_r = 360^\circ - A$  if A is given degrees  
and  
 $A_r = 2\pi - A$  if A is given in radians.



Reference Angle Values in Various Quadrants

Function	Quadrant		
	II	III	IV
$\sin \theta$	$\sin (180^\circ - \theta)$	$-\sin (\theta - 180^\circ)$	$-\sin (360^\circ - \theta)$
$\cos \theta$	$-\cos (180^\circ - \theta)$	$-\cos (\theta - 180^\circ)$	$\cos (360^\circ - \theta)$
$\tan \theta$	$-\tan (180^\circ - \theta)$	$\tan (\theta - 180^\circ)$	$-\tan (360^\circ - \theta)$
$\cot \theta$	$-\cot (180^\circ - \theta)$	$\cot (\theta - 180^\circ)$	$-\cot (360^\circ - \theta)$
$\sec \theta$	$-\sec (180^\circ - \theta)$	$-\sec (\theta - 180^\circ)$	$\sec (360^\circ - \theta)$
$\csc \theta$	$\csc (180^\circ - \theta)$	$-\csc (\theta - 180^\circ)$	$-\csc (360^\circ - \theta)$

**Rule 1:** using the definitions of the six trigonometric functions we can determine the values for  $(\pi - \theta)$  angles in terms of values of  $\theta$  angles and in summary we have:

$\sin(\pi - \theta) = \sin \theta$	$\cos(\pi - \theta) = -\cos \theta$	$\tan(\pi - \theta) = -\tan \theta$
$\cot(\pi - \theta) = -\cot \theta$	$\csc(\pi - \theta) = \csc \theta$	$\sec(\pi - \theta) = -\sec \theta$

**Rule 2:** (Formulas for Negatives) using the definitions of the six trigonometric functions we can determine the values for negative angles in terms of values of positive angles and in summary we have:

$\sin(-\theta) = -\sin \theta$	$\cos(-\theta) = \cos \theta$	$\tan(-\theta) = -\tan \theta$
$\cot(-\theta) = -\cot \theta$	$\csc(-\theta) = -\csc \theta$	$\sec(-\theta) = \sec \theta$

**Rule 3:** using the definitions of the six trigonometric functions we can determine the values for  $(\pi + \theta)$  angles in terms of values of positive  $\theta$  angles and in summary we have:

$\sin(\pi + \theta) = -\sin \theta$	$\cos(\pi + \theta) = -\cos \theta$	$\tan(\pi + \theta) = \tan \theta$
$\cot(\pi + \theta) = \cot \theta$	$\csc(\pi + \theta) = -\csc \theta$	$\sec(\pi + \theta) = -\sec \theta$

**Rule 4:** Identities expressing trig functions in terms of their complements:

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$
$\cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$	$\csc\left(\frac{\pi}{2} - \theta\right) = \sec \theta$	$\sec\left(\frac{\pi}{2} - \theta\right) = \csc \theta$

**Definition:** (Periodic Function) A function  $f$  is periodic if there exists a positive real number  $k$  such that  $f(\theta + k) = f(\theta)$  for every  $\theta$  in the domain of  $f$ . The least positive real number  $k$ , if it exists, is the period of  $f$ .

**Proposition:** (Periodic Function) the six trigonometric functions are periodic functions. Sine, cosine, secant, and cosecant have period  $2\pi$  while tangent and cotangent have period  $\pi$ .

$$\sin(\theta + 2\pi) = \sin \theta$$

$$\cos(\theta + 2\pi) = \cos \theta$$

$$\tan(\theta + \pi) = \tan \theta$$

**Definition:** (Even Function) A function of a real number  $x$  is called an even function when  $f(-x) = f(x)$  for all  $x$  in the domain of  $f$

**Definition:** (Odd Function) A function of a real number  $x$  is called an odd function when  $f(-x) = -f(x)$  for all  $x$  in the domain of  $f$

**Proposition:** (Parity of the Trigonometric Functions) the cosine and secant functions are even functions and the sine, cosecant, tangent and cotangent functions are odd functions.