Trigonometric Function Differentiation

(1) $f(x)=\sin x$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \qquad \sin(a+b)=\sin a\cos b+\cos a\sin b$$

$$=\lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h} \qquad \textit{Take } \sin x \textit{ as a common factor}$$

$$=(\sin x)\left[\lim_{h\to 0}\frac{\cos h-1}{h}\right]+(\cos x)\left[\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$=(\sin x)(0)+(\cos x)(1)=\cos x$$

$$f(x)=\sin x \Rightarrow f'(x)=\cos x$$

(2) $f(x)=\cos x$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \qquad \cos(a+b)=\cos a\cos b-\sin a\sin b$$

$$=\lim_{h\to 0}\frac{\cos x(\cos h-1)-\sin x\sin h}{h} \qquad \textit{Take } \cos x \textit{ as a common factor}$$

$$=(\cos x)\left[\lim_{h\to 0}\frac{\cos h-1}{h}\right]-(\sin x)\left[\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$=(\cos x)(0)-(\sin x)(1)=-\sin x$$

$$f(x)=\cos x \Rightarrow f'(x)=-\sin x$$

(3) $f(x)=\tan x=\dfrac{\sin x}{\cos x}$

$$f'(x)=\frac{(\cos x)(\cos x)-(\sin x)(-\sin x)}{(\cos x)^2}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x$$

$$f(x)=\tan x=\frac{\sin x}{\cos x}\Rightarrow f'(x)=\sec^2 x$$

(4) $f(x) = \sec x = \dfrac{1}{\cos x}$

$$f'(x) = \frac{(\cos x)(0) - 1(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

$$f(x) = \sec x = \frac{1}{\cos x} \Rightarrow f'(x) = \sec x \tan x$$

(5) $f(x) = \csc x = \dfrac{1}{\sin x}$

$$f'(x) = \frac{(\sin x)(0) - 1(\cos x)}{(\sin x)^2} = \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

$$f(x) = \csc x = \frac{1}{\sin x} \Rightarrow f'(x) = -\csc x \cot x$$

(6) $f(x) = \cot x = \dfrac{\cos x}{\sin x}$

$$f'(x) = \frac{(\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = \frac{-\cos^2 x - \sin^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$$

$$f(x) = \cot x = \frac{\cos x}{\sin x} \Rightarrow f'(x) = -\csc^2 x$$

The table below provides a summary for the basic trigonometric derivatives:

Function	Derivative
$f(x) = \sin x$	$f'(x) = \cos x$
$f(x) = \cos x$	$f'(x) = -\sin x$
$f(x) = \tan x$	$f'(x) = \sec^2 x$
$f(x) = \sec x$	$f'(x) = \sec x \tan x$
$f(x) = \cot x$	$f'(x) = -\csc^2 x$
$f(x) = \csc x$	$f'(x) = -\csc x \cot x$

Example 1: Find $\frac{d}{d\varphi}(\varphi - \sin\varphi)(1+\cos\varphi)$

Putting $u = \varphi - \sin\varphi, \quad v = 1+\cos\varphi$

Then $u' = 1 - \cos\varphi, \quad v' = -\sin\varphi$

From the formula $\frac{d}{d\varphi}(uv) = u'v + uv'$

$$\frac{d}{d\varphi}\left[(\varphi - \sin\varphi)(1+\cos\varphi)\right] = (1-\cos\varphi)(1+\cos\varphi) + (\varphi - \sin\varphi)(-\sin\varphi)$$
$$= 1 - \cos^2\varphi - \varphi\sin\varphi + \sin^2\varphi$$
$$= 2\sin^2\varphi - \varphi\sin\varphi \qquad \left[\text{using } 1\text{-}\cos^2\varphi = \sin^2\varphi\right]$$

Note: if you multiply the brackets out and then differentiate you get $\frac{d}{d\varphi}(\varphi - \sin\varphi)(1+\cos\varphi) = \frac{d}{d\varphi}(\varphi + \varphi\cos\varphi - \sin\varphi - \sin\varphi\cos\varphi)$ which involves using the product formula twice, on $\varphi\cos\varphi$ and $\sin\varphi\cos\varphi$

Example 2: Differentiate the function $r = \frac{\sin t}{t}$

$u = \sin t, \quad v = t \Rightarrow u' = \cos t, \quad v' = 1$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{\cos t \times t - \sin t \times 1}{t^2} = \frac{t\cos t - \sin t}{t^2}$$