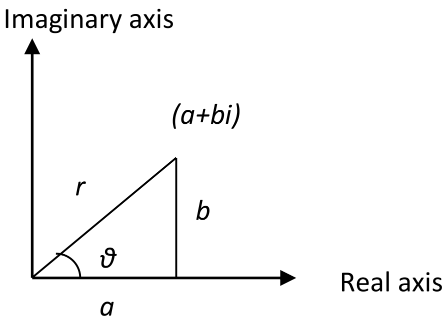
Trigonometric Form for Complex Numbers

A complex number is a number that can be represented in the form $a+bi$, where *a* is the real part and *bi* is the imaginary part.

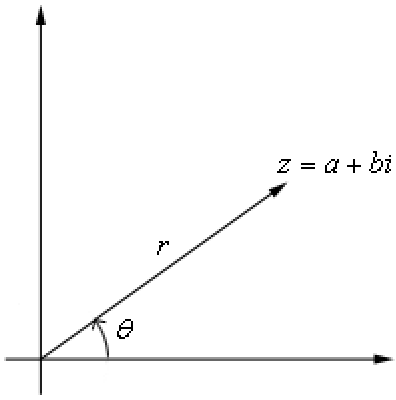
We can geometrically represent (i.e. graph) a complex number $a+bi$ in a plane using the ordered pair (*a*, *b*). The *x*-axis is relabeled as the **real axis** and the *y*-axis is relabeled as the **imaginary axis**.



The figure above correctly suggests that we can write a complex number using trigonometric functions.

We learned how to add, subtract, multiply and divide complex numbers. To work effectively with powers and roots of complex numbers, it is helpful to write complex numbers in trigonometric form.

Consider the nonzero complex number $a+bi$, let θ be the angle from the positive real axis(measured counterclockwise) to the segment connecting the origin and the point (a,b).



We can write: $a = r\cos\theta$ and $b = r\sin\theta$

where $r=\sqrt{a^2+b^2}$. So now we can define a complex number in trigonometric form: $z = a+bi = r\cos\theta + r\sin\theta i$

Definition 1: The trigonometric form of the complex number $z=a+bi$ is
$z = rcos\theta + r\sin\theta = r(\cos\theta + i\sin\theta)$

where $r=\sqrt{a^2+b^2}$ $\quad a = r\cos\theta$ $\quad b = r\sin\theta$ $\quad \tan\theta = \frac{b}{a}$

The number r is the **modulus of z** and θ is called the **argument of z** and is denoted by, $\theta = \arg z$

Remarks:

1) The trigonometric form of a complex number is also called the **polar form**. Sometimes the polar form will be written as, $z = |z|(\cos\theta + i\sin\theta)$

2) The abbreviation of $z = r(\cos\theta + i\sin\theta)$ is $z = rcis\theta$

There are infinitely many choices for θ, so the trigonometric form of a complex number is not unique. Normally, θ is restricted to the interval $0 \le \theta \le 2\pi$. The argument of *z* can be any of the infinite possible values of θ each of which can be found by solving $\tan\theta = \frac{b}{a}$ and making sure that θ is in the correct quadrant.

Rule 1: To write the complex number in trigonometric form:

Step1: Find $r = \sqrt{a^2 + b^2}$

Step 2: Find $\tan\theta = \frac{b}{a}$

Step 3: Make sure that θ is in the correct quadrant ($\tan\theta = \tan(\theta + \pi)$)

Example 1: Write the complex number $z = -2 - 2\sqrt{3}i$ in trigonometric form:

$$z = -2 - 2\sqrt{3}i \Rightarrow a = -2 \ \& \ b = -2\sqrt{3}$$

$$r = \sqrt{a^2 + b^2} = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{4 + 12} = \sqrt{16} = 4$$

$$\tan\theta = \frac{b}{a} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}$$

$$\Rightarrow \theta = \frac{\pi}{3} \text{ but } z \in QIII$$

$$\Rightarrow \theta = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$$

$$z = r(\cos\theta + i\sin\theta)$$

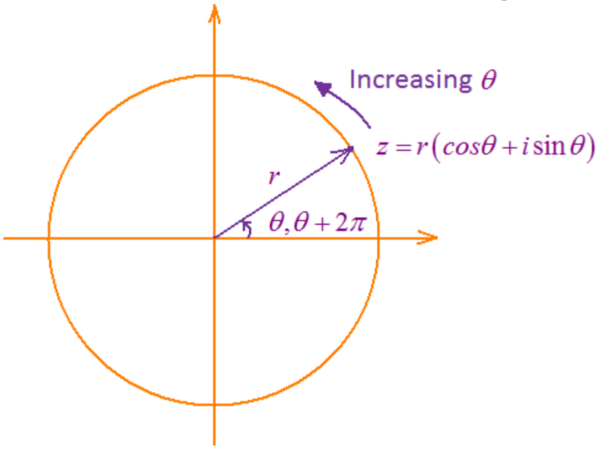
$$z = 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

The principle argument is the unique value of the argument that is in the range $-\pi < \arg \le \pi$ and is denoted by $Argz$.

Note that the inequalities at either end of the range tell that a negative real number will have a principle value of the argument of $Argz = \pi$.

If you now increase the value of θ, you will be just increasing the angle that the point makes with the positive *x*-axis by rotating the point about the origin in a counter-

Since it takes 2π radians to make one complete revolution you will be back at your initial starting point when you reach $\theta+2\pi$ and so have a new value of the argument. See the figure below.



If you keep increasing the angle you will again be back at the starting point when you reach $\theta+4\pi$, which is again a new value of the argument. Continuing in this fashion we can see that every time we reach a new value of the argument we will simply be adding multiples of 2π onto the original value of the argument.

Likewise, if you start at θ and decrease the angle you will be rotating the point about the origin in a clockwise manner and will return to your original starting point when you reach $\theta - 2\pi$.

Continuing in this fashion we can again see that each new value of the argument will be found by subtracting a multiple of 2π from the original value of the argument.

So we can see that if θ_1 and θ_2 are two values of arg *z* then for some integer *k* we will have,

$\theta_1-\theta_2=2\pi k$

So, as a conclusion any two values of the argument will differ from each other by a multiple of 2π .

$\arg z = Argz + 2\pi n.............n = 0,\pm1,\pm2,.....$

Remark: It is necessary to be careful in specifying the choices of $arc\tan\theta=\frac{y}{x}$ so that the point z lies in the appropriate quadrant.

$$\arg(z)=\begin{cases}\arctan\frac{y}{x} & \text{if } x>0, y\neq 0\\ \arctan\frac{y}{x}+\pi & \text{if } x<0, y\geq 0\\ \arctan\frac{y}{x}-\pi & \text{if } x<0, y<0\end{cases}$$

Rule 2: To write the complex number in standard form:

Step1: Find $a=r\cos\theta$
Step 2: Find $b=r\sin\theta$
Step 3: Simplify the result to have the form of $z=a+bi$