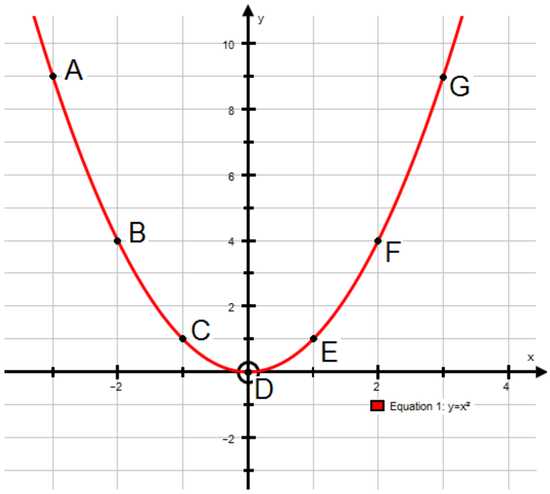
The Derivative

Consider the function $y = x^2$ between *x*=-3 and *x*=3

At the point A(-3,9) the value of *y* is seen to be decreasing rapidly as *x* increases. A **tangent** is a line which touches, but does not cross a curve. The **gradient** of a curve at any point P is defined as the slope of the tangent at P.

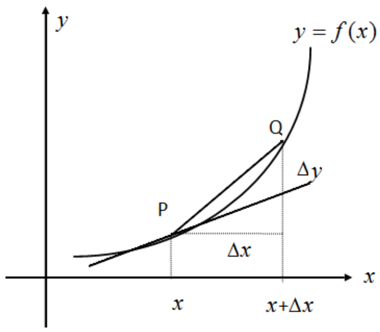


At A the graph has a large negative gradient. At the points B(-2,4) and C(-1,1) the gradient is still negative but is becoming less steep as we move from A through B to C. At the point D(0,0) the gradient is zero. Then at point E(1,1) there is a small positive gradient and this becomes larger as we move through F(2,4) to G(3,9).

Differentiation is the name given to the process by which an expression for the gradient of a function is obtained from the original function.

Definition 1: Derivative of a function: Let $y = f(x)$ be a function. The derivative of *f* at x, denoted by $f'(x)$, is defined to be

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



provided the limit exists. For all x for which this limit exists, f' is a function of x

The derivative of the function of x is also a function of x.

The process of finding the derivative of a function is called differentiation. A function is differentiable at x if its derivative exists at x and is differentiable on an open interval (a,b) if it is differentiable at every point in the interval.

In addition to $f'(x)$, which is read as "f prime of x", other notations are used to denote the derivative of $y = f(x)$ with respect to x. The most common are: y', $\frac{dy}{dx}$ and $\frac{d}{dx} f(x)$.

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

<u>Geometric Interpretation of the Derivative</u>

f' is the function whose value at *x* is the slope of the tangent line to the graph of *f* at *x*.

Definition 2: If f is defined on an open interval containing c, and if the limit

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

Exist, then the line passing through $(c, f(c))$ with slope m is the tangent line to the graph of f at the point $(c, f(c))$.

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the slope of the graph of f at x=c

Rate of Change Interpretation of the Derivative

If $y = f(x)$, then f' is the function whose value at *x* is the **instantaneous rate of change** of *y* at the point *x*.

Example 1: Use the definition of derivative to find the derivative of $f(x) = 2x$.

$$f(x+h) = 2(x+h)$$

$$\frac{f(x+h) - f(x)}{h} = \frac{2(x+h) - 2x}{h} = \frac{2h}{h}$$

$$\frac{f(x+h) - f(x)}{h} = 2$$

Since we end up here with a constant, taking the limit as $h \to 0$ won't make any difference:

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) = 2.$$

Theorem 1: If f is differentiable at x=c, then f is continuous at x=c

$$\textit{Differentiability} \Rightarrow \textit{Continuity}$$