The Chain Rule

So far we have been differentiating simple functions, like x^3, or $\cos(x)$ but most of the functions you are likely to encounter will be composite functions like $\left(3x^7+9x\right)^{11}$, $\cos(x^4)$ etc… The chain rule tells you how to differentiate such functions.

Composition and the generalized derivative rules

(1) $f(x)=(g\circ k)(x)=g(k(x))$

$$f'(x)=\lim_{b\to x}\frac{f(b)-f(x)}{b-x}$$

$$=\lim_{b\to x}\frac{g(k(b))-g(k(x))}{b-x}$$

$$=\lim_{b\to x}\frac{g(k(b))-g(k(x))}{b-x}\bullet\frac{k(b)-k(x)}{k(b)-k(x)}$$

$$=\lim_{b\to x}\frac{g(k(b))-g(k(x))}{k(b)-k(x)}\bullet\lim_{b\to x}\frac{k(b)-k(x)}{b-x}$$

$$=\lim_{k(b)\to k(x)}\frac{g(k(b))-g(k(x))}{k(b)-k(x)}\bullet\lim_{b\to x}\frac{k(b)-k(x)}{b-x}$$

$$=g'(k(x))\bullet k'(x)$$

This derivative rule for the composition of functions is called the Chain Rule.

Rule: The Chain Rule: If $y=f(x)$ is a differentiable function of u and $u=g(x)$ is a differentiable function of x, then $y=f(g(x))$ is a differentiable function of x and:

$$\frac{dy}{dx}=\frac{dy}{du}\bullet\frac{du}{dx}$$

or equivalently,

$$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)\cdot g'(x)$$

In words, the derivative of a composite function is the product of the derivatives of the outside and inside functions. The derivative of the outside function must be evaluated at the inside function.

The product and chain rules can be used to differentiate a quotient, rather than using the quotient rule. The resulting formulas may look different, but they will be equivalent.

(2) $f(x) = g(k(x))$ where $g(x) = x^n$.

Then $f(x) = [k(x)]^n$

$g(x) = x^n \Rightarrow g'(x) = nx^{n-1} \Rightarrow g'(k(x)) = n[k(x)]^{n-1}$

Thus, $f'(x) = g'(k(x)) \cdot k'(x) = n[k(x)]^{n-1} \cdot k'(x)$

This derivative rule for the power of a function is called the Generalized Power Rule.

(3) $f(x) = g(k(x))$ where $g(x) = \sin x$.

$f(x) = \sin[k(x)]$

$g(x) = \sin x \Rightarrow g'(x) = \cos x \Rightarrow g'(k(x)) = \cos[k(x)]$

Thus, $f'(x) = g'(k(x)) \cdot k'(x) = \cos[k(x)] \cdot k'(x)$.

Similarly for all the other formulas and here is a summary for all the formulas

Function	Derivative
$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$	$f(x) = u^n \Rightarrow f'(x) = nu^{n-1} \bullet u'$
$f(x) = \sin x \Rightarrow f'(x) = \cos x$	$f(x) = \sin[k(x)] \Rightarrow f'(x) = \cos[k(x)] \cdot k'(x)$
$f(x) = \cos x \Rightarrow f'(x) = -\sin x$	$f(x) = \cos[k(x)] \Rightarrow f'(x) = -\sin[k(x)] \cdot k'(x)$
$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$	$f(x) = \tan[k(x)] \Rightarrow f'(x) = \sec^2[k(x)] \cdot k'(x)$
$f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$	$f(x) = \sec[k(x)] \Rightarrow f'(x) = \sec[k(x)] \tan[k(x)] \cdot k'(x)$
$f(x) = \cot x \Rightarrow f'(x) = -\csc^2 x$	$f(x) = \cot[k(x)] \Rightarrow f'(x) = -\csc^2[k(x)] \cdot k'(x)$
$f(x) = \csc x \Rightarrow f'(x) = -\csc x \cot x$	$f(x) = \csc[k(x)] \Rightarrow f'(x) = -\csc[k(x)] \cot[k(x)] \cdot k'(x)$

Example 1: Suppose *f* and *g* are differentiable functions such that:

$f(1) = 9$	$f(2) = -5$	$g(1) = 2$	$g(9) = 3$
$f'(1) = -2$	$f'(2) = -6$	$g'(1) = 4$	$g'(9) = 7$

Find each of the following:

1) $(f\circ g)'(1)$

$(f\circ g)'(1)=f'(g(1))\cdot g'(1)=f'(2)\cdot g'(1)=(-6)(4)=-24$

2) $(g\circ f)'(1)$

$(g\circ f)'(1)=g'(f(1))\cdot f'(1)=g'(9)\cdot f'(1)=7(-2)=-14$

3) $h'(1)$ if $h(x)=\sqrt{f(x)}$

$h(x)=\sqrt{f(x)}=[f(x)]^{1/2}$

$\Rightarrow h'(x)=\frac{1}{2}[f(x)]^{-1/2}\cdot f'(x)=\dfrac{f'(x)}{2\sqrt{f(x)}}$

$\Rightarrow h'(1)=\dfrac{f'(1)}{2\sqrt{f(1)}}=\dfrac{-2}{2\sqrt{9}}=-\dfrac{1}{3}$

4) $j'(1)$ if $j(x)=[g(x)]^5$

$j(x)=[g(x)]^5$

$\Rightarrow j'(x)=5[g(x)]^4\cdot g'(x)$

$\Rightarrow j'(1)=5[g(1)]^4\cdot g'(1)=5(2)^4(4)=320$

5) $l'(1)$ if $l(x)=\dfrac{3}{[f(x)]^2}$

$l(x)=\dfrac{3}{[f(x)]^2}=3[f(x)]^{-2}$

$\Rightarrow l'(x)=-6[f(x)]^{-3}\cdot f'(x)$

$\Rightarrow l'(1)=\dfrac{-6f'(1)}{[f(1)]^3}=\dfrac{-6(-2)}{9^3}=\dfrac{12}{729}=\dfrac{4}{243}$

6) $s'(1)$ if $s(x)=\sin[f(x)]$

$s'(x)=\cos[f(x)]\cdot f'(x)$

$\Rightarrow s'(1)=\cos[f(1)]\cdot f'(1)=\cos(9)\cdot(-2)=-2\cos 9$

7) $m'(1)$ if $m(x)=\sec[g(x)]$

$m'(x)=\sec[g(x)]\tan[g(x)]\cdot g'(x)$

$\Rightarrow m'(1)=\sec[g(1)]\tan[g(1)]\cdot g'(1)=\sec(2)\tan(2)\cdot 4=4\sec 2\tan 2$

WHICH COME FIRST?

To help decide which comes first, think of how you would calculate the value of the function if you were given a numerical value for the variable. If you had the function $y = \ln(\sin x)$, you would find the value of sinx first and then take the ln of the result. Hence in using the chain rule to differentiate the function you put $u = \sin x$. The table below gives some further examples.

Function	$\sin 6x$	$\sqrt{x^2+1}$	$\cos^4 x$	$(3x^2+x-5)^4$	e^{3x+4}
First (u)	$6x$	x^2+1	$\cos x$	$3x^2+x-5$	$3x+4$
Second	sin	root	power	power	exp