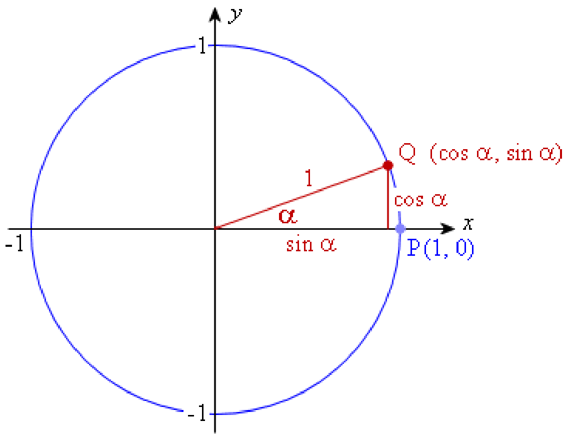
Sum and Difference Formulas

We will prove the cosine of the sum of two angles identity first, and then show that this result can be extended to all the other identities given.
cos (α+β) = cos α cos β − sin α sin β

We draw a circle with radius 1 unit, with point *P* on the circumference at (1, 0).
We draw an angle α from the centre with terminal point *Q* at (cos α, sin α), as shown. [*Q* is (cos α, sin α) because the hypotenuse is 1 unit.]

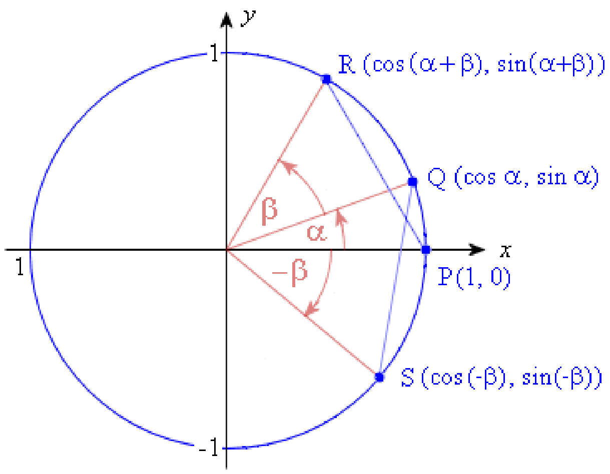


We extend this idea by drawing:

The angle β with terminal points at *Q* (cos α, sin α) and *R* (cos (α+β), sin (α+β))

The angle -β with terminal point at *S* (cos (-β), sin (-β))

The lines *PR* and *QS* are equal in length.



Now, using the distance formula from analytical geometry, we have:

$$PR^2 = \left(\cos(\alpha+\beta)-1\right)^2 + \sin^2(\alpha+\beta)$$

$$PR^2 = \cos^2(\alpha+\beta) - 2\cos(\alpha+\beta) + 1 + \sin^2(\alpha+\beta) \qquad \left[\cos^2(\alpha+\beta)+\sin^2(\alpha+\beta)=1\right]$$

$$PR^2 = 2 - 2\cos(\alpha+\beta)$$

Now using the distance formula on distance QS:

$$QS^2 = \left(\cos\alpha - \cos(-\beta)\right)^2 + \left(\sin\alpha - \sin(-\beta)\right)^2$$
$$QS^2 = \cos^2\alpha - 2\cos\alpha\cos(-\beta) + \cos(-\beta)^2 + \sin^2\alpha - 2\sin\alpha\sin(-\beta) + \sin^2(-\beta)$$
$$QS^2 = 2 - 2\cos\alpha\cos(-\beta) - 2\sin\alpha\sin(-\beta) \qquad \left[\cos^2\theta + \sin^2\theta = 1\right]$$
$$QS^2 = 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta \qquad \left[\cos\beta = \cos(-\beta)\ \&\ \sin(-\beta) = \sin\beta\right]$$

Since $PR = QS$, we can equate the 2 distances we just found:

$$2 - 2\cos(\alpha+\beta) = 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$$

Subtracting 2 from both sides and dividing throughout by -2, we obtain:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

If we replace β with (-β), this identity becomes:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad \left[\cos\beta = \cos(-\beta)\ \&\ \sin(-\beta) = \sin\beta\right]$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

In the sine formulas, + or − on the left is also + or − on the right. But in the cosine formulas, + on the left becomes − on the right; and vice-versa.

Rule 1: The **sine** of the sum and difference of two angles is as follows:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Rule 2: The **cosine** of the sum and difference of two angles is as follows:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

We can verify all the reference angles formulas using the sum and difference formulas.

Example 1: Let: $\alpha = \frac{\pi}{2}$ and $\beta = \beta$, verify that $\cos\left(\frac{\pi}{2}+\beta\right) = -\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos\left(\frac{\pi}{2}+\beta\right) = \cos\frac{\pi}{2}\cos\beta - \sin\frac{\pi}{2}\sin\beta$

$\Rightarrow \cos\left(\frac{\pi}{2}+\beta\right) = \cancel{(0)\cos\beta} - (1)\sin\beta$

$\Rightarrow \cos\left(\frac{\pi}{2}+\beta\right) = -\sin\beta$

$\cos\left(\frac{\pi}{2}+\beta\right) = -\sin\beta$

Rule 3: The **tangent** of the sum and difference of two angles is as follows:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$