Solving Exponential & Logarithmic Functions

There are two methods for solving exponential and logarithmic equations. One method is fairly simple, but requires a very special form of the exponential and logarithmic equation. The other will work on more complicated exponential equations, but can be a little messy at times.
Let's start off by looking at the simpler method.

$b^x = b^y \Leftrightarrow x = y$ **The bases must be the same. If it isn't then this fact will not help.**

$\log_a x = \log_a y \Leftrightarrow x = y$ **The bases must be the same and the coefficients must be 1.**

Example 1: Solve: $5^{3x} = 5^{7x-2}$

The two bases are the same $\Rightarrow$ the rule $b^x = b^y \Leftrightarrow x = y$ is applicable.

$5^{3x} = 5^{7x-2}$
$\Rightarrow 3x = 7x - 2$
$\Rightarrow -4x = -2$
$\Rightarrow x = \frac{-2}{-4} = \frac{1}{2}$

Another method depends upon the Inverse Properties that we saw earlier $a^{\log_a x} = a$ and $\log_a a^x = x$

Example 2: Solve:

1) $\ln x - \ln 7 = 0$

$\ln x - \ln 7 = 0$
$\Rightarrow \ln x = \ln 7$
$\Rightarrow e^{\ln x} = e^{\ln 7}$
$\Rightarrow x = 7$

2) $\left(\frac{1}{4}\right)^x = 64$

$\left(\frac{1}{4}\right)^x = 64$
$\Rightarrow 4^{-x} = 4^3$
$\Rightarrow -x = 3$
$\Rightarrow x = -3$

3) $e^x = 3$

$e^x = 3$
$\Rightarrow \ln e^x = \ln 3$
$\Rightarrow x = \ln 3$

4) $\log_2 x = 3$

$\log_2 x = 3$
$\Rightarrow 2^{\log_2 x} = 2^3$
$\Rightarrow x = 8$

Strategies for solving exponential and logarithmic equations

1) Rewrite the original equation in a form that allows the use of the One-to-One properties of exponential or logarithmic functions.
2) Rewrite an exponential equation in logarithmic form and apply the Inverse Property of logarithmic functions
3) Rewrite an logarithmic equation in exponential form and apply the Inverse Property of exponential functions

Example 3: Solve: $2\left(3^{2t-4}\right)-5=13$

$$2\left(3^{2t-4}\right)-5=13$$
$$\Rightarrow 3^{2t-4}=\frac{13+5}{2}=9$$
$$\Rightarrow 3^{2t-4}=9$$
$$\Rightarrow \log_3 3^{2t-4}=\log_3 9=\log_3 3^2=2$$
$$\Rightarrow 2t-4=2$$
$$\Rightarrow 2t=6$$
$$\Rightarrow t=3$$

Example 4: Solve $\ln(2x+1)=\ln 3-\ln x$

Exponentiate both sides, using *e* as the base: $e^{\ln(2x+1)}=e^{\ln 3-\ln x}$.

Since $a^{x-y}=\frac{a^x}{a^y}$, we can write the right hand side as $e^{\ln 3-\ln x}=\frac{e^{\ln 3}}{e^{\ln x}}=\frac{3}{x}$.

Since the left hand side is 2*x* + 1, we get the equation $2x+1=\frac{3}{x}$ or $2x^2+x=3$.

This is a quadratic equation, so we can set things equal to zero and get:

$$2x^2+x-3=0$$
$$(2x+3)(x-1)=0.$$

The roots are *x* = 1 and *x* = – 3/2.
x = – 3/2 doesn't satisfy the original equation, so *x* = 1 is the only solution.

Example 5: Solve $2^{x+1}=9^x$.

$\ln 2^{x+1}=\ln 9^x$.
We then use property of logarithms to bring both variables down out of the exponents.

$$\Rightarrow (x+1)\ln 2=x\ln 9$$
$$\Rightarrow x\ln 2+\ln 2=x\ln 9$$
$$\Rightarrow \ln 2=x\ln 9-x\ln 2=x(\ln 9-\ln 2)$$

Therefore, $x=\frac{\ln 2}{\ln 9-\ln 2}\approx 0.4608$.

Rule 1: Changing Logarithmic Bases: For any logarithmic bases *a* and *b*, and any positive number *m*,
$\log_b M = \dfrac{\log_a M}{\log_a b}$.

Note: In particular $\log_b M = \dfrac{\ln M}{\ln b}$.

Example 6: Solve $\log_2(3x+1) = 2$.

Since the problem uses a log base 2, we will exponentiate both sides, using 2 as the base.

$$2^{\log_2(3x+1)} = 2^2 = 4$$

Using the key fact, we get $3x+1=4$, so *x* = 1.

Example 7: Solve the equation $\log_2 x = \log_2 9$
According to the fourth property $\log_a x = \log_a y \Leftrightarrow x = y$ we conclude that:
$\log_2 x = \log_2 9 \Leftrightarrow x = 9$

Example 8: Solve $2\ln(x) = \ln(2+x)$.

The base is *e*. However, when we do exponentiation the left hand side of the equation, we can't use the key fact, since the expression $e^{2\ln(x)}$ does not have "*e*" and "ln" adjacent.
However, we can use the property we can write: $2\ln(x) = \ln x^2$.
Now the equation is $\ln(x^2) = \ln(2+x)$. When we do exponentiation both sides, we get

$$x^2 = 2 + x$$
$$x^2 - x - 2 = 0$$

Solving the quadratic we find that *x* = – 1 and *x* = 2.
However, *x* = – 1 is not a solution to the original equation, since the domain of $\ln(x)$ is all strictly positive numbers.

Example 9: Solve $3^{2x+1} - 28\cdot 3^x + 9 = 0$.

Note that the equation is equivalent to $3\cdot 3^{2x} - 28\cdot 3^x + 9 = 0$.

Let $y = 3^x$, then the equation becomes

$$3y^2 - 28y + 9 = 0$$
$$(y-9)(3y-1) = 0$$
$$y = 9 \text{ or } \frac{1}{3}$$
$$3^x = 9 \text{ or } \frac{1}{3}$$
$$x = 2 \text{ or } -1$$