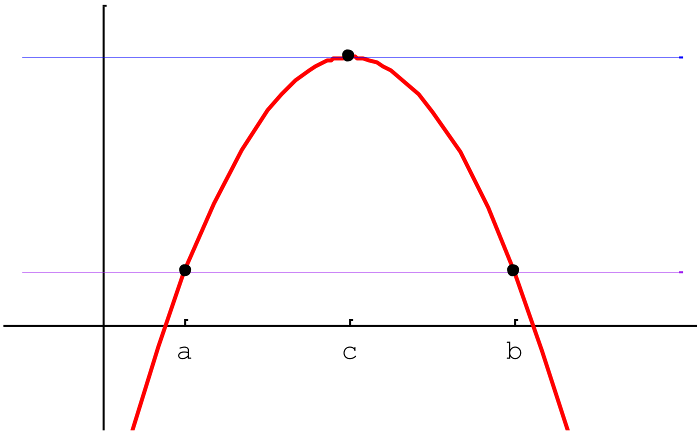
# Rolle's Theorem and Mean Value Theorem

**Rolle's Theorem:**

Consider the graph given below. Note that at the points *x* = *a* and *x* = *b* we have *f*(*a*) = *f*(*b*). Thus, the points (*a*, *f*(*a*)) and (*b*, *f*(*b*)) not only lie on the graph of *y* = *f*(*x*), but they also lie on the same horizontal line. Notice also, that there is a point *x* = *c* where the line tangent to the graph is horizontal. That is, $f'(c)=0$.




**Theorem 1:** **Rolle's Theorem**: Suppose that *y* = *f*(*x*) is continuous at every point of the closed interval [*a*, *b*] and differentiable at every point of its interior (*a*, *b*). If *f*(*a*) = *f*(*b*), then there is at least one number *c* in (*a*, *b*) such that $f'(c)=0$.

**Note 1:** We can have more than one point where $f'(c)=0$ on the interval [*a*, *b*].

**Proof:** Proof of Rolle's Theorem: Let $f(a)=d=f(b)$.

**Case 1:** If $f(x)=d\ \forall\ x\in[a,b]$, $f$ is a constant on the interval and $f'(x)=0\ \forall\ x\in(a,b)$.

**Case 2:** Suppose $\exists\ x\in[a,b]\ni f(x)>d$. By the Extreme Value Theorem, you know that $f$ has a maximum at some *c* in the interval. Since $f(c)>d$, this maximum does not occur at either endpoint. So $f$ has a maximum in the *open* interval $(a,b)$. This implies that $f(c)$ is a relative maximum and thus a critical number of $f$. Finally, because $f$ is differentiable at *c*, you can conclude that $f'(c)=0$.

**Case 3:** Suppose $\exists\ x\in[a,b]\ni f(x)<d$. By the Extreme Value Theorem, you know that $f$ has a minimum at some *c* in the interval. Since $f(c)<d$, this minimum does not occur at either endpoint. So $f$ has a minimum in the *open* interval $(a,b)$. This implies that $f(c)$ is a relative minimum and thus a critical number of $f$. Finally, because $f$ is differentiable at *c*, you can conclude that $f'(c)=0$.

Remark 1: Remember that *y* = *f*(*x*) must be continuous on the closed interval [*a*, *b*] and differentiable on the open interval (*a*, *b*), and that *f*(*a*) = *f*(*b*) in order to apply Rolle's Theorem. If we fail any of the conditions, we are NOT guaranteed at least one point where $f'(c)=0$

Example 1: Find the two *x*-intercepts of the function $f(x)=-3x\sqrt{x+1}$ and show that $f'(x)=0$ at some point between the two *x*-intercepts.

$f(x)=-3x\sqrt{x+1}$ is continuous at every point of the closed interval [-1, $+\infty$] and differentiable at every point of its interior (-1, $+\infty$)

$$f(x)=-3x\sqrt{x+1}$$

$$0=-3x\sqrt{x+1}$$

$$0=-3x \Rightarrow x=0$$

$$or$$

$$0=\sqrt{x+1} \Rightarrow x=-1$$

$$x-\text{intercepts: } (0,0) \text{ or } (-1,0)$$

$$f'(x)=-3x\left[\frac{1}{2}(x+1)^{-1/2}\right]+(x+1)^{1/2}(-3)$$

$$=-3(x+1)^{-1/2}\left[\frac{x}{2}+(x+1)\right]$$

$$=-3(x+1)^{-1/2}\left(\frac{3}{2}x+1\right)$$

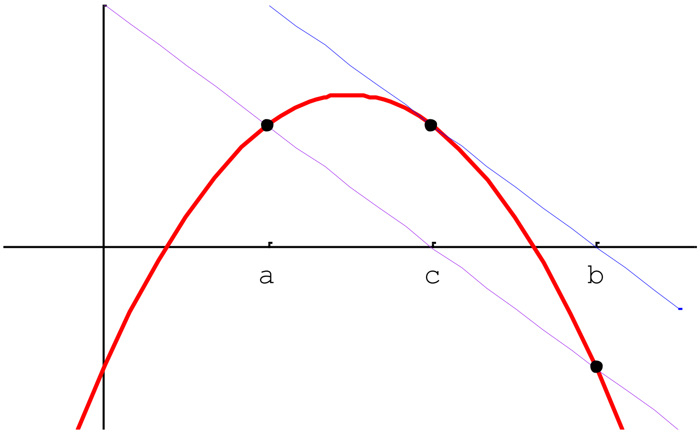
$$0=-3(x+1)^{-1/2}\left(\frac{3}{2}x+1\right)$$

$$x=-1 \text{ or } x=-\frac{2}{3}$$

$$\text{but } x=-1 \text{ is not included on } (-1,0), \text{ so } x=-\frac{2}{3}$$

**<u>The Mean Value Theorem</u>**

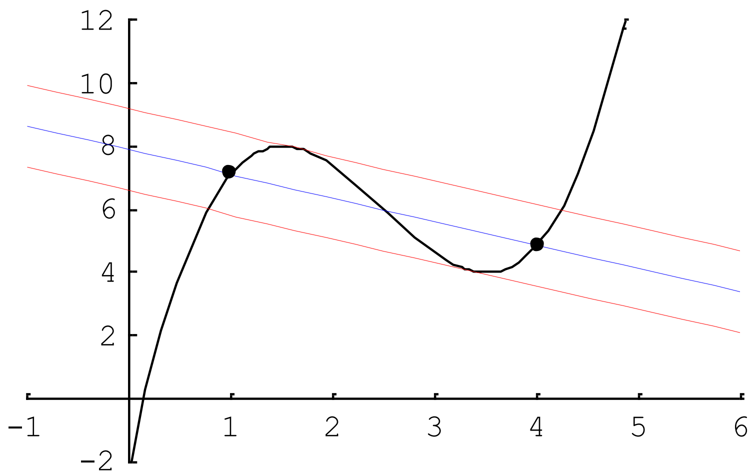
The Mean Value Theorem is a more general case of Rolle's Theorem. We remove the requirement that *f*(*a*) = *f*(*b*). We still can draw a line through the points (*a*, *f*(*a*)) and (*b*, *f*(*b*)), but the line is no longer horizontal. But there is a point *x* = *c* where the tangent line to *y* = *f*(*x*) at c is parallel to the line through (*a*, *f*(*a*)) and (*b*, *f*(*b*)).




Theorem 2: **The Mean Value Theorem**: If $f$ is continuous on a closed interval $[a,b]$ and differentiable on the open interval $(a,b)$, then there exists at least one number $c$ in $(a,b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Another way to write the result of the Mean Value Theorem is: $f(b)-f(a)=f'(c)(b-a)$

Note 2: In the Mean Value Theorem, "there is at least one point" means there could be more than one point. For instance, consider the function *y* = *f*(*x*) whose graph is given below.




Notice that on the interval [1, 4] there are two points where $f'(c)=\dfrac{f(4)-f(1)}{4-1}$.

**<u>To check if the Mean Value theorem is applicable:</u>**

1) Find $f'(c)=\dfrac{f(b)-f(a)}{b-a}$
2) Find the derivative of the function $f'(x)$
3) Equate $f'(x)$ to $f'(c)$
4) Solve for x