Rational Functions

Rational Expressions are fractions where the numerator and denominator are polynomials. It has the form of $f(x)=\frac{polynomial}{polynomial}=\frac{p(x)}{q(x)}$, where the bottom polynomial, *q*(*x*), can not equal zero. However, since the polynomial in the denominator can have zeros, whenever it does have zeros the function is undefined. This brings us to a new idea. At the values that give a zero in the bottom polynomial *q*(*x*), we draw vertical lines at these values which are called **vertical asymptotes**. The function will approach the vertical asymptote(s), but **<u>the function never crosses or touches the vertical asymptote(s)</u>**. The function will either go off into the positive infinity or negative infinity direction as it approaches the vertical asymptote(s). We will learn how to determine the direction a function diverges to as it approaches a vertical asymptote.

<u>Finding vertical asymptotes</u>

For every value that makes the denominator equal to zero, *x* equal to that value is a vertical asymptote. So at each of those values of *x*, we draw a dotted vertical line.

Example 1: Find the vertical asymptotes of $f(x)=\frac{1}{x^2-25}$

In order to find the vertical asymptotes, we have to solve the quadratic equation $x^2-25=0$ Notice that this can be factored as a difference of squares, $x^2-25=(x+5)(x-5)=0$. So we have $f(x)=\frac{1}{(x+5)(x-5)}$, where the denominator is zero at the two values *x* = 5 and *x* = -5. Hence we have two vertical asymptotes for this function. To graph the vertical asymptotes, we draw dotted vertical lines at *x* = 5 and *x* = -5.

Example 2: Does the function $f(x)=\frac{x^2+2x+1}{x+1}$ have any vertical asymptotes?

At first glance you might say that *x* = -1 gives a vertical asymptote. However, if you factor the numerator the function becomes $f(x)=\frac{x^2+2x+1}{x+1}=\frac{(x+1)(x+1)}{x+1}=x+1$ which is just a linear function. Hence, after simplification, this function has no vertical asymptotes. Although this function has what is called a **removable vertical asymptote** at the vertical line *x* = -1, the value *x* = -1 **is not** in the domain of the original function. So the graph of this function after simplification is a straight line with a hole in it at the value *x* = -1.

<u>Finding horizontal asymptotes</u>

Rational functions also have **horizontal asymptotes**. Horizontal asymptotes are horizontal lines that the graph gets close to as *x* tends toward infinity. That is, as the values of *x* grow, then if the function approaches the line *y* = **b**, **b** a real number, then the line *y* = **b** is a horizontal asymptote. However there is one big difference between horizontal and vertical asymptotes, **<u>a function can cross a horizontal asymptote</u>**. This occurs only when you can set the function equal to the value **b**, and you are able to solve for a value of *x*. If this is possible, then the function will cross the horizontal asymptote at the point (*x*, **b**).

Let f be the rational function given by

$$f(x)=\frac{a_n x^n + a_{n-1}x^{n-1}+\cdots+a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1}+\cdots+b_1 x + b_0}, \quad a_n \neq 0, b_m \neq 0.$$

The degree of the numerator is *n*. The degree of the denominator is *m*.

1) If $n<m$, the *x*-axis, or $y=0$, is the horizontal asymptote of the graph of f.
2) If $n=m$, the line $y=\frac{a_n}{b_m}$ is the horizontal asymptote of the graph of f.
3) If $n>m$, the graph of f has no horizontal asymptote.

Finding oblique asymptote

There is one more case we need to consider. If the degree of the polynomial in the numerator is one more than the degree of the polynomial in the denominator, then we have an **oblique asymptote**. These are non-vertical, non-horizontal asymptotes that the graph gets close to. These asymptotes are slanted lines. To find them you must divide the bottom polynomial into the top polynomial by using **long division**. Long division is the type of division we use to divide polynomials by polynomials.

Example 3: Find the oblique asymptote of the rational function

$$f(x)=\frac{x^2+2x-1}{x-3}$$

Degree of the numerator is greater than the degree of the denominator, we will use long division to find the oblique asymptote. The result will be *y* equals to a linear equation which will be the oblique asymptote.

Here we do the following:

$$\begin{array}{r l}
 & x+5 \\
x-3 \,\big) & x^2+2x-1 \\
 & -\left(x^2-3x\right) \\
 & \quad 5x-1 \\
 & \quad -\left(5x-15\right) \\
 & \qquad 14
\end{array}$$

Example 4: Find the vertical, horizontal and oblique asymptotes if possible of:

$$y = \frac{5x^2 + 1}{3x^2 + 5x - 2}$$

- Degree of the numerator=Degree of denominator

  $\Rightarrow y = \frac{5}{3}$ is a horizontal asymptote
- To find the vertical asymptote we set the denominator equal to zero

  $3x^2 + 5x - 2 = 0$

  $\Rightarrow (3x-1)(x+2) = 0$

  $\Rightarrow x = \frac{1}{3}$ & $x = -2$

  $\Rightarrow x = \frac{1}{3}$ and $x = -2$ are vertical asymptotes

Example 5: Find all asymptotes of

$$f(x) = \frac{2x^2 - 3x - 1}{x - 2}$$

- Since $x - 2 = 0 \Rightarrow x = 2$, this gives us a vertical asymptote at the line $x = 2$.
- There are no horizontal asymptotes since the degree of the numerator is greater than the degree of the denominator.
- Dividing: $(2x^2 - 3x - 1) \div (x - 2)$, we get $(2x+1) + \frac{1}{x-2}$. Now we see that when

  $x \to \infty$ or $x \to -\infty$, $\frac{1}{x-2} \to 0$ and the value of $f(x) \to 2x+1$. This means that as the absolute value of x becomes very large, the graph of $f(x)$ gets very close to the graph of $y = 2x+1$. Thus the line $y = 2x+1$ is the oblique asymptote.