Product – to – Sum Formulas

The other three product-to-sum formulas come from the other three ways to add or subtract the sum and difference formulas. If you subtract the two cosine formulas instead of adding:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

You get:

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta$$
$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$

If we add the two identities instead, we have:

$$\cos(x - y) + \cos(x + y) = \cos x\cos y + \cos x\cos y = 2\cos x\cos y$$

$$\Rightarrow \cos x\cos y = \frac{1}{2}\left(\cos(x - y) + \cos(x + y)\right)$$

Recall the identities $\sin(x + y) = \sin x\cos y + \cos x\sin y$ and $\sin(x - y) = \sin x\cos y - \cos x\sin y$. If we add the two identities, we have:

$$\sin(x + y) + \sin(x - y) = \sin x\cos y + \sin x\cos y = 2\sin x\cos y$$

$$\Rightarrow \sin x\cos y = \frac{1}{2}\left(\sin(x + y) - \sin(x - y)\right)$$

If instead, we subtract the second identity from the first, we have:

$$\sin(x + y) - \sin(x - y) = \cos x\sin y + \cos x\sin y = 2\cos x\sin y$$

$$\Rightarrow \cos x\sin y = \frac{1}{2}\left(\sin(x + y) - \sin(x - y)\right)$$

To get the other product – to sum formulas, add the two sine or subtract them. Here are all four formulas together:

Rule 1: **De-factorization (Products into Sums or Differences)**

1) $\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$

2) $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha + \beta) + \{\cos(\alpha - \beta)\right]$

3) $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha + \beta) + \sin(\alpha - \beta)\right]$

4) $\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha + \beta) - \sin(\alpha - \beta)\right]$

The fourth one of those formulas really isn't needed, because you can always evaluate $\cos\alpha\sin\beta$ as $\sin\alpha\cos\beta$. But it's traditional to present all four formulas.

Example 1: Write the expression as a sum of trigonometric functions

1) sin(6θ)sin(4θ)
sin(6θ)sin(4θ)=½ [sin(6θ-4θ) –cos(6θ+4θ)= ½ [cos(2θ)-cos(10θ)]

2) cos(3θ)cosθ
cos(3θ)cosθ =½ [cos(3θ-θ)+ cos(3θ+θ)]= ½ [cos(2θ) + cos(4θ)]

3) sin(3θ)cos(5θ)
sin(3θ)cos(5θ)=½ [sin(3θ+θ) + sin(3θ-θ)]= 1/2 [sin(4θ) + sin(2θ)