Polynomials and Synthetic Division

Definition 1: A polynomial of degree n is a function of the form

$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_2 x^2 + ax + a_0$ where the $a's$ are real numbers (called the coefficients of the polynomial) and $a_n \neq 0$

The degree of a polynomial function is the highest power of x in its expression. Constant (non-zero) polynomials, linear polynomials, quadratic and cubic polynomials are polynomials of degree 0, 1, 2, 3 and 4 respectively.

$f(x) = a_2 x^2 + ax + a_0, where a_2 \neq 0$ is a quadratic polynomial function.

$f(x) = ax + a_0, where a \neq 0$ is a linear polynomial function.

$f(x) = a_0$ is a quadratic polynomial function.

Rule 1: If $P(x)$ and $D(x)$ are polynomials such that $D(x) \neq 0$, and the degree of $D(x)$ is less than or equal to the degree of $P(x)$, there exist unique polynomials $Q(x)$ and $R(x)$ such that:

$$\underset{\substack{\uparrow \\ Dividend}}{P(x)} = \underset{\substack{\uparrow \\ Divisor}}{D(x)} \times \underset{\substack{\uparrow \\ Quotient}}{Q(x)} + \underset{\substack{\uparrow \\ Remainder}}{R(x)}$$

Where $R(x) = 0$ or the degree of $R(x)$ is less than the degree of $D(x)$. If the remainder $R(x)$ is zero, $D(x)$ divides evenly into $P(x)$.

The division algorithm can also be written as: $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$

Long Division of Polynomials

Long division of polynomials is a lot like long division of real numbers. If the polynomials involved were written in fraction form, the numerator would be the dividend, and the denominator would be the divisor.

To divide polynomials using long division:

- Divide the first term of the dividend by the first term of the divisor. This is the first term of the quotient.
- Multiply the new term by the divisor.
- Subtract this product from the dividend. This difference is the new dividend.
- Repeat these steps, using the difference as the new dividend until the first term of the divisor is of a greater degree than the new dividend.

The last "new dividend" whose degree is less than that of the divisor is the remainder.
If the remainder is zero, the divisor divided evenly into the dividend.

Example 1: $f(x)=2x^3-4x^2+x+2$ is divided by $g(x)=x^2+4$

$$\begin{array}{r|l} & 2x-4 \\ \hline x^2+4 & 2x^3-4x^2+x+2 \\ & \underline{2x^3+8x} \\ & 0-4x^2-7x+2 \\ & \underline{....-4x^2-16} \\ & ....-7x+18 \end{array}$$

$$\left(2x^3-4x^2+x+2\right)\div\left(x^2+4\right)=2x-4+\frac{-7x+18}{x^2+4}$$

Remark: $16\div 3=\frac{16}{3}=5+\frac{1}{3}$ is similar to $\left(2x^3-4x^2+x+2\right)\div\left(x^2+4\right)=2x-4+\frac{-7x+18}{x^2+4}$

<u>Synthetic division</u>

Synthetic division is a "shortcut" method of performing polynomial long division. We can use synthetic division when we want to divide a polynomial f(x) by a linear divisor of the form $x-a$, where the coefficient of x is 1. Since *a* can be positive, negative or zero, the divisor can look like:

- x
- $x-a$
- $x+a$

Those are the only cases we can use synthetic division. If we are not dividing by a linear divisor of the form $x-a$, then we need to use polynomial long division.

Synthetic division requires only three lines:

1) The top line for the dividend and divisor
2) The second line for the intermediate values
3) The third line for the quotient and remainder

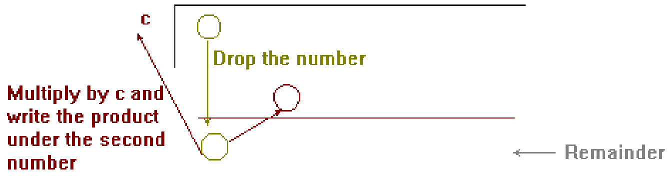
Let the dividend have degree n, to divide two polynomials using synthetic division follow the steps listed below:

1) In line one, write the coefficients of the polynomial as the dividend, and let c be the divisor. (The coefficients should be for x when the polynomial function is arranged in descending order.)

2) In line three rewrite the leading coefficient of the dividend directly below its position in the dividend; multiply it by the divisor, and write the product in line two directly below the coefficient of x^{n-1}.

3) Add this product to the number directly above it in the dividend (this number is the coefficient of x^{n-1}) to get a new number.
Repeat steps two and three until the entire polynomial has been divided.

The quotient will be one degree less than the dividend.
The coefficients of the quotient are the first n - 1 numbers in line three.
The remainder is the last number in line three.



We can use synthetic division to

- determine if x = a is a zero of a polynomial f(x)
- determine whether ($x-a$) is a factor of f(x)
- evaluate f(a)

Theorem 1: **The Remainder Theorem:** If a polynomial $f(x)$ is divided by $(x-k)$, then the remainder is $r=f(k)$.

Extending this theorem, we will obtain:

If a polynomial $f(x)$ is divided by $(sx-t)$, then the remainder is $r=f\left(\frac{t}{s}\right)$.

Rule 2: **When there is no remainder as a result of synthetic division i.e. if f is a polynomial function and *a* is a real number, the following statements are equivalent**

- x = a is a zero of $f(x)$
- f(a) = 0
- (x - a) is a factor of $f(x)$
- the graph has an x-intercept at the point (a, 0)

Rule 3: **When there is a non-zero remainder r (r $\neq$ 0) as a result of synthetic division i.e. if f is a polynomial function and *a* is a real number, the following statements are equivalent**

- f(a) = r$\neq$ 0
- (x - a) is not a factor of $f(x)$
- x = a is not a zero of $f(x)$
- the graph does not have an x-intercept at the point (a, 0)