Polynomial Functions of Higher Degrees

If a is a zero of a polynomial and the exponent on the term that produced the root is k then we say that a has **multiplicity** k.
Zeroes with a multiplicity of 1 are called simple zeroes.

Definition 1: A factor $(x-a)^k, k>1$ yields a repeated zero $x=a$ of multiplicity k.

Example 1: The polynomial $h(x)=(x+1)^3(x-1)^2(x+5)(x-4)$ have four zeroes:

1) $x=-1$ *of* multiplicity 3
2) $x=1$ *of* multiplicity 2
3) $x=5$ *of* multiplicity 1
4) $x=4$ *of* multiplicity 1

Real Zeros of a Polynomial Functions

If f is a polynomial function and a is a real number, the following statements are equivalent.

- $x=a$ is a zero of the function f.
- $x=a$ is a solution of the polynomial equation $f(x)=0$
- $(x-a)$ is a factor of the polynomial $f(x)$
- $(a,0)$ is an x-intercept of the graph of f.

To locate the real zeros we use the intermediate value theorem which gives the expected number of zeros in a certain interval. This theorem depends on the signs of $f(a)$ and $f(b)$

Theorem 1: **Intermediate Value Theorem:** If $f(x)$ is a polynomial with only real coefficients, and if for real numbers a and b, the values $f(a)$ and $f(b)$ are opposite in sign, then there exists at least one real zero between a and b.

Example 2: Does $h(x)=4x^3-x^2+x-5$ have any real zeros between 1 and 4?

$h(1)=4(1)^3-(1)^2+(1)-5=-1$

$h(4)=4(4)^3-(4)^2+(4)-5=239$

$h(1)=-1<0$ and $h(4)=239>0$, according to the Intermediate Value Theorem there must be a real zero between 1 and 4

Variations in Sign: The number of variations in sign of $f(x)$ is given by the number of "sign flips" as the nonzero coefficients $f(x)$ are read from left to right in the standard form.

Example 3: Find the number of variations in sign in $f(x) = 7x^6 - 2x^3 + 4x + 5$

Because the leading coefficient is positive, we may want to clearly place a + sign in front of it:

$$f(x) = +7x^6 - 2x^3 + 4x + 5$$

There are 2 variations in sign. (Don't worry about "missing terms"; they have 0 coefficients.)

Parity: Two integers have the same parity $\Leftrightarrow$ they are both even or both odd.

Descartes' Rule of Signs: Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2 x^2 + ax + a_0$ where the $a's$ are real numbers (called the coefficients of the polynomial) and $a_n \neq 0$

Let:

z^+ is the number of positive real zeros of $f(x)$

z^- is the number of negative real zeros of $f(x)$

m is the number of variations in sign in $f(x)$

n is the number of variations in sign in $f(-x)$

Then, $0 \le z^+ \le m$, where z^+ has the same parity as m

and $0 \le z^- \le n$, where z^- has the same parity as n

Example 4: Based on Descartes' Rule of Signs, give the possible number of positive and negative zeros of $f(x) = 2x^5 - 8x^4 + 3x^3 + 19x - 23$

$$f(x) = 2x^5 - 8x^4 + 3x^3 + 19x - 23$$

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$\Rightarrow$ The number of variations in sign in $f(x)$ is 3

$\Rightarrow$The possible number of positive zeros is 1 or 3.

$$f(-x) = 2(-x)^5 - 8(-x)^4 + 3(-x)^3 + 19(-x) - 23$$

$$f(-x) = -2x^5 - 8x^4 - 3x^3 - 19x - 23$$

$$f(-x) = -2x^5 - 8x^4 - 3x^3 - 19x - 23$$

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$\Rightarrow$ The number of variations in sign in $f(-x)$ is 0

$\Rightarrow$The possible number of positive zeros is 0.