Partial Decomposition

In the Division Algorithm, the rational expression $\frac{f(x)}{d(x)}$ is:

1) **Improper** if the degree of $f(x)$ is greater than $d(x)$.
2) **Proper** if the degree of $f(x)$ is less than $d(x)$.

Partial Decomposition of Improper Rational Expressions
To find the partial fractions follow the steps listed below:

Step 1: Use long division of polynomials to get a quotient $p(x)$ and a remainder $r(x)$. Then write:

$R(x)=\frac{P(x)}{Q(x)}=p(x)+\frac{r(x)}{Q(x)}$, where the degree of $r(x)$ is less than that of $Q(x)$.

Step 2: Factor the denominator $Q(x)=q_1(x)q_2(x).....q_n(x)$, where each factor $q_i(x)$ is either linear $ax+b$, or irreducible quadratic ax^2+bx+c, or a power of the form $(ax+b)^n$ or $(ax^2+bx+c)^n$.

Step 3: Decompose $\frac{r(x)}{Q(x)}$ into partial fractions of the form:

$\frac{r(x)}{Q(x)}=F_1(x)+F_2(x)+.F_3(x)+......$, where each fraction is of the form

$F_i(x)=\frac{A}{(ax+b)^k}$ Or $F_i(x)=\frac{Ax+B}{(ax^2+bx+c)^k}$

where $1\le k\le n$ (n is the exponent of $ax+b$ or ax^2+bx+c in the factorization of $Q(x)$.)

Example 1: Decompose $R(x)=\frac{x^3+x^2+2}{x^2-1}$ into partial fraction:

Using long division, $R(x)=\frac{x^3+x^2+2}{x^2-1}=x+1+\frac{x+3}{x^2-1}$

Now we need to decompose $\frac{x+3}{x^2-1}$. First, we need to factorize x^2-1.

$a^2-b^2=(a-b)(a+b)$

$x^2-1=(x-1)(x+1)$

$$\frac{x+3}{x^2-1}=\frac{x+3}{(x-1)(x+1)}=\frac{A}{x+1}+\frac{B}{x-1}$$

$$\frac{A}{x+1}+\frac{B}{x-1}=\frac{A(x-1)+B(x+1)}{(x-1)(x+1)}$$

$$\frac{A(x-1)+B(x+1)}{(x-1)(x+1)}=\frac{(A+B)x+(-A+B)}{(x-1)(x+1)}$$

$$\frac{(A+B)x+(-A+B)}{(x-1)(x+1)}\equiv\frac{x+3}{(x-1)(x+1)}$$

$$\begin{cases}(A+B)x\equiv x\\(-A+B)\equiv 3\end{cases}\Leftrightarrow\begin{cases}A+B=1\\-A+B=3\end{cases}$$

Therefore, B = 2 and A = -1

$$\frac{x+3}{x^2-1}=\frac{-1}{x+1}+\frac{2}{x-1}$$

$$R(x)=\frac{x^3+x^2+2}{x^2-1}=x+1+\frac{-1}{x+1}+\frac{2}{x-1}$$

Partial Decomposition of Proper Rational Expressions

In this case we skip the first step and we start directly with the second step.

Example 2: Decompose $R(x)=\frac{x+2}{x^2-x-2}$ into partial fractions:

We need to decompose $\frac{x+2}{x^2-x-2}$. First, we need to factorize x^2-x-2. $x^2-x-2=(x-2)(x+1)$

$$\frac{x+2}{x^2-x-2}=\frac{x+2}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$$

$$\frac{A}{x-2}+\frac{B}{x+1}=\frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

$$\frac{A(x+1)+B(x-2)}{(x-2)(x+1)}=\frac{Ax+A+Bx-2B}{(x-2)(x+1)}=\frac{(A+B)x+(A-2B)}{(x-2)(x+1)}$$

$$\frac{(A+B)x+(A-2B)}{(x-2)(x+1)} \equiv \frac{x+2}{(x-2)(x+1)}$$

$$\begin{cases}(A+B)x \equiv x \\ (A-2B) \equiv 2\end{cases} \Leftrightarrow \begin{cases}A+B=1 \\ A-2B=2\end{cases}$$

Therefore, $B=\frac{-1}{3}$ and $A=\frac{4}{3}$

$$\frac{x+2}{x^2-x-2}=\frac{\frac{4}{3}}{x-2}+\frac{\frac{-1}{3}}{x+1}$$

$$\frac{x+2}{x^2-x-2}=\frac{4}{3(x-2)}+\frac{-1}{3(x+1)}$$

$$R(x)=\frac{x+2}{x^2-x-2}=\frac{4}{3(x-2)}+\frac{-1}{3(x+1)}$$

Rule 1: Assume that $Q(x)$ is in the factored form and the degree of $r(x)$ is less than degree of $Q(x)$ .Then $\frac{r(x)}{Q(x)}$ is decomposed into partial fractions in the following way:

(1) For each factor of the form $(x-r)^k$ write $\frac{A_1}{(x-r)}+\frac{A_2}{(x-r)^2}+.\frac{A_3}{(x-r)^3}+.....+\frac{A_k}{(x-r)^k}$ where $A_1, A_2, A_3, ...A_k$ are coefficients to be determined.

(2) For each factor of the form $(ax^2+bx+c)^k$ write

$$\frac{B_1x+C_1}{(ax^2+bx+c)^1}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+.\frac{B_3x+C_3}{(ax^2+bx+c)^3}+.....+\frac{B_kx+C_k}{(ax^2+bx+c)^k}$$

, where $B_1, B_2, B_3, ...B_k$ and $C_1, C_2, C_3, ...C_k$ are coefficients to be determined.

Example 3: Decompose $R(x)=\frac{4x^5-2x^4+2x^3-8x^2-2x-3}{(x-1)^2(x^2+x+1)^2}$ into partial fractions:

The denominator is already factored, so the four partial fractions are:

$$\frac{4x^5-2x^4+2x^3-8x^2-2x-3}{(x-1)^2(x^2+x+1)^2}=\frac{A}{(x-1)^1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{(x^2+x+1)^1}+\frac{Ex+F}{(x^2+x+1)^2}$$