Operations with Complex Numbers

A complex number is of the form $a+bi$, where a is called the real part and bi is called the imaginary part. When performing operations involving complex numbers, we will be able to use many of the techniques we use with polynomials.

Addition and Subtraction of Complex Numbers

When adding complex numbers, we are only allowed to add real parts to other real parts, and imaginary parts to other imaginary parts.

Definition 1: The addition of two complex numbers a + b i and c + d i is defined as follows.

(a + b i) + (c + d i) = (a + c) + (b + d) i

Definition 2: The subtraction of two complex numbers a + b i and c + d i is defined as follows.

(a + b i) - (c + d i) = (a - c) + (b - d) i

Example 1: Find the sum: $(7+3i)+(5+2i)$

Add the real parts together.

$7+5=12$

Add the imaginary parts together.

$3i+2i=5i$

The solution is: $12+5i$

Multiplication of Complex Numbers

Multiplying complex numbers works like multiplying two binomials.

Definition 3: The multiplication of two complex numbers a + bi and c + di is defined as follows:

(a + b i)(c + d i) = (a c - b d) + (a d + bc) i

<u>Division of Complex Numbers</u>

We use the multiplication property of complex number and its conjugate to divide two complex numbers.

Earlier, we learned how to rationalize the denominator of an expression like: $\frac{3}{2+\sqrt{5}}$

We multiplied numerator and denominator by the conjugate of the denominator, $2+\sqrt{5}$:

$$\frac{3}{2+\sqrt{5}}\times\frac{2-\sqrt{5}}{2-\sqrt{5}}=\frac{3\left(2-\sqrt{5}\right)}{\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)}=\frac{6-3\sqrt{5}}{4-5}=-1\left(6-3\sqrt{5}\right)=3\sqrt{5}-6$$

We did this so that we would be left with no radical (square root) in the denominator. Dividing with complex numbers is similar.

The division of two complex numbers, $\frac{z_1}{z_2}=\frac{a+bi}{c+di}$ can be thought of as simply a process for eliminating the *i* from the denominator and writing the result as a new complex number $a+bi$.

Complex Conjugate

The conjugate of a complex number $z = a + bi$ is a complex number $\bar{z} = a - bi$

Properties: The following properties of the complex conjugate are easy to verify:

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
2. $\overline{-z} = -\bar{z}$
3. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
4. $\overline{z_1 \bullet z_2} = \overline{z_1} \bullet \overline{z_2}$
5. $\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$
6. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
7. z is real if and only if $\bar{z} = z$

With the standard convention that the real and imaginary parts are denoted by Re z and Im z, we have:

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \ \operatorname{Im} z = \frac{z - \bar{z}}{2i}$$

If $z = a + bi$, then $z\bar{z} = a^2 + b^2$

If $z = a + bi$, the modulus of z is the non-negative real number $|z|$ defined by $|z| = \sqrt{a^2 + b^2}$.

Geometrically, the modulus of z is the distance from z to 0.

More generally, $|z_1 - z_2|$ is the distance between z_1 and z_2 in the complex plane. For:

$$\begin{aligned}
&|z_1 - z_2| \\
&= |(a_1 + ib_1) - (a_2 + ib_2)| \\
&= |(a_1 - a_2) + i(b_1 - b_2)| \\
&= \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}
\end{aligned}$$

Properties: The following properties of the modulus are easy to verify, using the identity:

1. $|z|^2 = z\bar{z}$
2. $|z_1 \bullet z_2| = |z_1| \bullet |z_2|$
3. $|z^{-1}| = |z|^{-1}$
4. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

There is a very nice relationship between the modulus of a complex number and it's conjugate. Let's start with a complex number $z = a + bi$ and take a look at the following product

$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2$

From this product we can see that $z\bar{z} = |z|^2$

Notice as well that in computing the modulus the signs of the real and imaginary parts of the complex number won't affect the value of the modulus and so we can also see that,

$|\bar{z}| = |z|$

$|-z| = |z|$

Theorem: The triangle inequality theorem: The modulus of the sum of two complex numbers is less than or equal to the sum of their modulus

$|z_1 + z_2| \leq |z_1| + |z_2|$