Operations with Complex Numbers

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. To find the product, we have:

$$z_1 \bullet z_2 = \left[r_1(\cos\theta_1 + i\sin\theta_1)\right]\left[r_2(\cos\theta_2 + i\sin\theta_2)\right]$$
$$= r_1 \bullet r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 \bullet r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\sin\theta_1 i\sin\theta_2)$$
$$= r_1 \bullet r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$
$$= r_1 \bullet r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_1\cos\theta_2)\right]$$
$$= r_1 \bullet r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) = r_1 \bullet r_2\, cis(\theta_1+\theta_2)$$

Rule 1: If the trigonometric forms of two complex numbers z_1 and z_2 are $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then to **multiply** two complex numbers, we **multiply** the moduli and **add** the arguments

$$z_1 \bullet z_2 = r_1 \bullet r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

Example 1: Let $z_1 = 3(\cos 50 + i\sin 50)$ and $z_2 = 4(\cos 10 + i\sin 10)$ find $z_1 z_2$

$$z_1 z_2 = 3(\cos 50 + i\sin 50) \bullet 4(\cos 10 + i\sin 10)$$
$$z_1 z_2 = 3 \bullet 4\left[\cos(50+10) + i\sin(50+10)\right]$$
$$z_1 z_2 = 12\left[\cos 60 + i\sin 60\right]$$

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. To find the quotient, we have:

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$
$$= r_1(\cos\theta_1 + i\sin\theta_1) \bullet \frac{1}{r_2(\cos\theta_2 + i\sin\theta_2)}$$
$$= r_1(\cos\theta_1 + i\sin\theta_1) \bullet \frac{1}{r_2}(\cos(-\theta_2) + i\sin(-\theta_2))$$
$$= r_1 \bullet \frac{1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos(-\theta_2) + i\sin(-\theta_2))$$
$$= \frac{r_1}{r_2}(\cos\theta_1\cos(-\theta_2) + i\sin\theta_1\cos(-\theta_2) + i\sin\theta_1\cos(-\theta_2) + i\sin\theta_1 i\sin(-\theta_2))$$
$$= \frac{r_1}{r_2}((\cos\theta_1\cos(-\theta_2) + i\sin\theta_1\cos(-\theta_2)) + i(\sin\theta_1\cos(-\theta_2) - \sin\theta_1\sin(-\theta_2)))$$
$$= \frac{r_1}{r_2}\left[(\cos\theta_1\cos(-\theta_2) - \sin\theta_1\sin(-\theta_2)) + i(\sin\theta_1\cos(-\theta_2) + \sin\theta_1\cos(-\theta_2))\right]$$
$$= \frac{r_1}{r_2}(\cos(\theta_1 + (-\theta_2)) + i\sin(\theta_1 + (-\theta_2)))$$
$$= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$
$$= \frac{r_1}{r_2} cis(\theta_1 - \theta_2)$$

Rule 2: If the trigonometric forms of two complex numbers z_1 and z_2 are $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then to divide two complex numbers, we divide the moduli and subtract the arguments

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right], \text{ for } z_2 \neq 0$$