Newton's Method

We cannot always solve for the exact value of a solution to an equation. If this is the case, we will need numerical methods to approximate the solution.

Procedure for Newton's Method:
Suppose we wish to approximate a solution to the equation $f(x) = 0$.
Graphically, this means that we want the x-intercept of the graph of $y = f(x)$.
The idea behind Newton's Method is to approximate the root (or zero, or solution) by a sequence of numbers. The limit of the sequence is the desired root.

Let $y = f(x)$ have a root, and let x_0 be an approximation of this root (ie, x_0 is close to the desired root).
Find the equation of the line tangent to f at x_0.

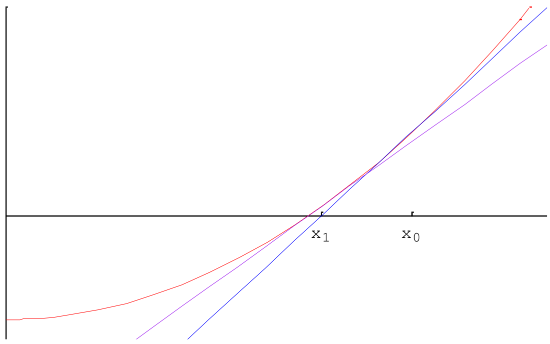
$$y - f(x_0) = f'(x_0)(x - x_0) \quad \text{or}$$
$$y = f'(x_0)(x - x_0) + f(x_0)$$

$y = f'(x_0)(x - x_0) + f(x_0)$

Notice that the x-intercept of the tangent line is closer to the root than x_0 is. Solve for the x-intercept.

Now, find the equation of the line tangent to f at x_1.

$$y - f(x_1) = f'(x_1)(x - x_1) \quad \text{or} \quad y = f'(x_1)(x - x_1) + f(x_1)$$

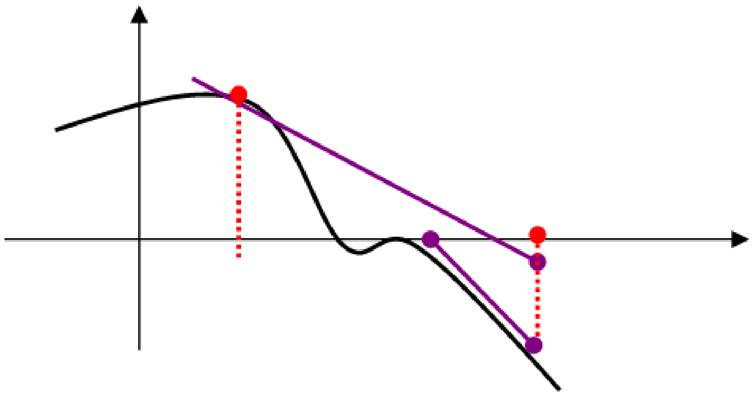


Notice that the x-intercept of this tangent line is closer to the root than x_0 or x_1. Solve for the x-intercept.

$y = f'(x_1)(x - x_1) + f(x_1)$

The sequence of numbers x_0, x_1, x_2, . . . has a limit, and that limit is the root.

Newton's Method uses tangent lines to approximate zeros. The assumption is that the function and the tangent line cross the x-axis at about the same point



NEWTON'S METHOD FOR APPROXIMATING THE ZEROS OF A FUNCTION

Let $f(c)=0$ where f is differentiable on an open interval containing c. Then, to approximate c, use the following steps.

Step 1: Make an initial estimate x_1 that is close to c.

Step 2: Determine a new approximation $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$.

Step 3: If $|x_n-x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to step 2 and calculate a new approximation.

Each successive application of this procedure is called an **iteration**.

Newton's Method is used to approximate a zero of a function

$c-\dfrac{f(c)}{f'(c)}$ *where c is the 1st approximation*

Example 1: If Newton's Method is used to approximate the real root of $x^3 + x - 1 = 0$, then a first approximation of $x_1 = 1$ would lead to a *third* approximation of x_3:

$$f(x)=x^3+x-1$$
$$f'(x)=3x^2+1$$

$$1-\frac{f(1)}{f'(1)}=\frac{3}{4} \quad or \quad .750=x_2$$

$$\frac{3}{4}-\frac{f(\frac{3}{4})}{f'(\frac{3}{4})}=\frac{59}{86}\ or\ .686=x_3$$