Logarithmic Functions

Logarithms were developed in the 17th century by the Scottish mathematician, John Napier. They were a clever method of reducing long multiplications into much simpler additions (and reducing divisions into subtractions). Young Johnny Napier had to help his dad, who was a tax collector. Johnny got sick of multiplying and dividing large numbers all day and devised logarithms to make his life easier.
The use of logarithms made trigonometry and many other fields of mathematics much simpler to calculate.
When **calculus** was developed later in the century, logarithms became central to many solutions. Today, logarithms are still important in many fields of science and engineering, even though we use calculators for most simple calculations.

Definition 1: The function given by $f(x) = \log_a x$ is called the logarithmic function with base a.
The equations $y = \log_a x$ and $x = a^y$ are equivalent.

$$y = \log_a x \Leftrightarrow x = a^y \qquad a > 0 \,,\; x > 0 \text{ and } a \neq 1$$

$y = \log_a x$ is in **logarithmic** form and the $x = a^y$ is in **exponential** form.

When evaluating logarithms, remember that a logarithm is an exponent. This means that $\log_a x$ is the exponent y to which a must be raised to obtain x.

For example, we know that the exponential equation $3^2 = 9$ is true.
In this case, the **base** is 3 and the **exponent** is 2. We can write this equation in **logarithm form**
$\log_3 9 = 2$
We say this as "the logarithm of 9 to the base 3 is 2".

Example 1: Find the domain and range of :

$$\log_3\left[\frac{x^2-3x-10}{x+2}\right]$$

g(x) = $\log_3\left[\frac{x^2-3x-10}{x+2}\right]$

To find the domain of this function, we must solve the rational inequality

$$\frac{x^2-3x-10}{x+2} > 0$$

Since $x^2 - 3x - 10 = (x+2)(x-5)$ we have

$$\frac{x^2-3x-10}{x+2} = \frac{(x+2)(x-5)}{x+2} = x-5 \quad (\text{ if } x \neq -2)$$

Thus we may solve the simpler inequality

$$x-5 > 0 \quad x > 5$$

The domain of g(x) is $(5, \infty)$. The range is the same as that of f (x) = log x, $(-\infty, \infty)$

Properties of Logarithms

1) $\log_a 1 = 0 \text{ because } a^0 = 1$	2) $\log_a a = 1 \text{ because } a^1 = a$
3) $\log_a a^x = x \,\&\, a^{\log_a x} = x$ (Inverse Property)	4) $\log_a x = \log_a y \Leftrightarrow x = y$ (One–to–One Property)

Example 2: Evaluate each function for the given value of x.

$f(x) = \log_2 x;\ x = 32$ and $x = 16$

1) $f(32) = \log_2 32 = \log_2 2^5 = 5$ (Using the property $\log_a a^x = x$)

2) $f(16) = \log_2 16 = \log_2 2^4 = 4$ (Using the property $\log_a a^x = x$)

Example 3: Simplify the expression $6^{\log_6 23}$

According to the third property $\log_a a^x = x \,\&\, a^{\log_a x} = x$ we conclude that:

$6^{\log_6 23} = 23$

The graphs of all logarithmic functions have similar characteristics. To graph any logarithmic function we have to:

- Specify the domain and the range
- Check for symmetry, shifting, reflecting, and translation
- Construct the table of values for $x \,\&\, y$
- Find the asymptote
- Draw a Cartesian system
- Plot and join the points

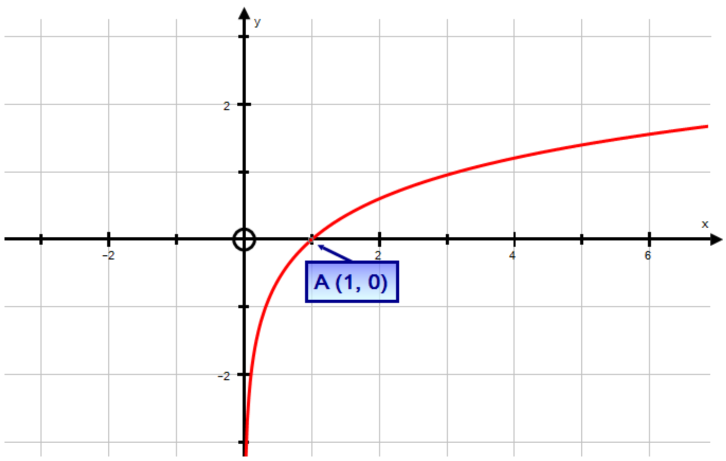
Example 4: Graph of the function $f(x) = \log_2 x$ is shown:

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

Intercept: $(1, 0)$

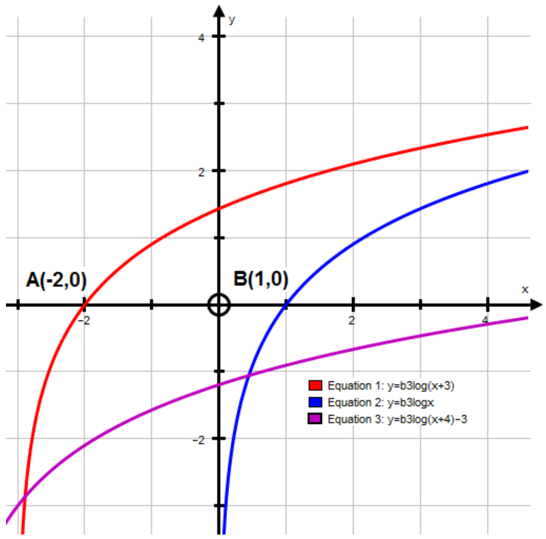
Asymptote: x=0



The graph of $\log_a x$ is used to sketch the graph of functions of the form $y = b \pm \log_b(x + c)$, using horizontal and vertical shifts.

Example 5: Graph $g\left(x\right)=\log_3 x$, $f\left(x\right)=\log_3(x+3)$ and $h\left(x\right)=\log_3(x+4)-3$

- $f\left(x\right)=\log_3(x+3)$ is obtained by shifting $g\left(x\right)=\log_3 x$ horizontally 3 units to the left
- $h\left(x\right)=\log_3(x+4)-3$ is obtained by shifting $g\left(x\right)=\log_3 x$ horizontally 4 units to the left and vertically 3 units downwards



Property: Let a be a positive number such that $a\neq 1$, and let n be a real number. If u and v are positive real numbers, the following properties are true.

1) $\log_a\left(uv\right)=\log_a u+\log_a v$
2) $\log_a \dfrac{u}{v}=\log_a u-\log_a v$
3) $\log_a u^n = n\log_a u$

Note: A logarithmic function without a base is of base 10: $f\left(x\right)=\log x \Leftrightarrow f\left(x\right)=\log_{10} x$

Example 6: Write as a single logarithmic function:

a) $2\log_3(3x)+\dfrac{1}{2}\log_3(25x^4)-\dfrac{2}{3}\log_3(64x^6)-3\log_3(2x^2)$

$\log_3(3x)^2+\log_3(25x^4)^{\frac{1}{2}}-\log_3(64x^6)^{\frac{2}{3}}-\log_3(2x^2)^3$

$\log_3(3x)^2+\log_3[\ (5x^2)^2]^{\frac{1}{2}}-\log_3[\ (4x^2)^3\ ]^{\frac{2}{3}}-\log_3(2x^2)^3$

$= \log_3(9x^2)+\log_3(5x^2)-\log_3(16x^4)-\log_3(8x^6)$

$= \log_3\left[\dfrac{(9x^2).(5x^2)}{(16x^4)(8x^6)}\right]=\log_3\left(\dfrac{45}{128x^6}\right)$

b) $\dfrac{1}{3}\left(\log_y p^3+\log_y q^2\right)-\dfrac{1}{2}\left(\log_y q^{\frac{4}{3}}\right)$ as a single logarithm.

$\dfrac{1}{3}\left(\log_y p^3+\log_y q^2\right)-\dfrac{1}{2}\left(\log_y q^{\frac{4}{3}}\right)$

$= \log_y (p^3)^{1/3} + \log_y(q^2)^{1/3} - \log_y (q^{4/3})^{½}$

$= \log_y p + \log_y q^{2/3} - \log_y q^{2/3}$

$= \log_y p$

Example 7: Simplify: $\log_a x + 3\log_a(x + 1) - 2\log_a x$.

$\log_a x + 3\log_a(x + 1) - 2\log_a x$

$= \log_a x + \log_a(x + 1)^3 - \log_a x^2$ power law of logarithms

$= \log_a x(x + 1)^3 - \log_a x^2$ product law of logarithms

$= \log_a \frac{x(x+1)^3}{x^2}$

Example 8: Given that $\log_5 2 = a$ and $\log_5 3 = b$, express the following in terms of a and b:

1) $\log_5 6$
   $\log_5 6 = \log_5 2 + \log_5 3 = a + b$

2) $\log_5 0.75$
   $\log_5 0.75 = \log_5 3 - \log_5 4 = \log_5 3 - 2\log_5 2 = b - 2a$

3) $\log_5 24$
   $\log_5 24 = \log_5 3 + \log_5 8 = \log_5 3 + 3\log_5 2 = 3a + b$