Limits

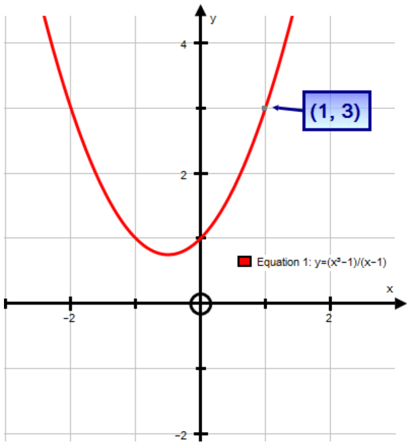
Suppose you are asked o sketch the graph of the function f given by: $f(x)=\frac{x^3-1}{x-1}; x\neq 1$

For all values other than $x=1$, you can use standard curve sketching techniques. However, at $x=1$, it is not clear what to expect. To get an idea of the behavior of the graph of f near $x=1$, you can use two sets of x-values – one set that approaches 1 from the left and one set approaches 1 from the right.

x approaches 1 from left → ← x approaches 1 from right

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	2.71	2.97	2.997	?	3.003	3.31	3.813

f(x) approaches 3 → ← f(x) approaches 3



The graph has a gap at the point (1,3). Although x cannot be equal to 1 but you can arbitrarily close to 1, and as a result $f(x)$ arbitrarily close to 3. Using the limit notation, you can write:

$\lim_{x\to 1} f(x)=3$ This is read as "the limit of $f(x)$ as x approaches 1 is 3"

The limit of the function $f(x)$ as x approaches c is the number L. *This limit is written as:*

$$\lim_{x\to c} f(x)=L$$

Limits and Function Values

If the limit of a function f as x approaches c exists, this limit may not be equal to $f(c)$. In fact, $f(c)$ may not even be defined.

The existence or nonexistence of $f(x)$ at $x=c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.

Theorem 1: **The Existence of a Limit:** Let f be a function and let c and L be real numbers, The limit of $f(x)$ as x approaches c is L if and only if: $\lim_{x\to a^+} f(x)=L$ and $\lim_{x\to a^-} f(x)=L$

$$\lim_{x\to a} f(x)=L \Leftrightarrow \lim_{x\to a^+} f(x)=L \text{ and } \lim_{x\to a^-} f(x)=L$$

Nonexistence of Limits

The limit of a function f as x approaches c **may fail to exist if**:

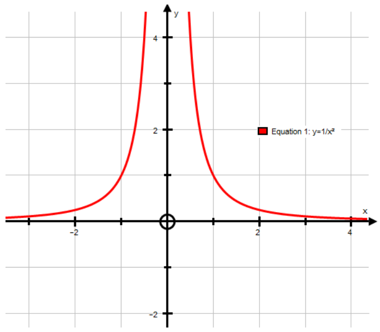
- $f(x)$ becomes infinitely large or infinitely small as x approaches c from either side.

- $f(x)$ approaches L as x approaches c from the right and $f(x)$ approaches M, $M \neq L$, as x approaches c from the left.

- $f(x)$ oscillates infinitely many times between two numbers as x approaches c from either side.

Example 1: **A Function that Approaches Infinity**: Consider the function $f(x)=\frac{1}{x^2}$, the

$\lim_{x\to 0}\frac{1}{x^2}$ does it exist?



As you can see from the graph x approaches 0 from either the left or the right, $f(x)$ increases without bound.

$f(x)$ is not approaching a real number L as x approaches 0,
Therefore you can conclude that the limit does not exist.

PART 1: **Evaluating Limits Analytically**

Theorem 2: **Some Basic Limits:**
Limit of a Constant: If d is a constant, then $\lim_{x\to c} d = d$.

Limit of the Identity Function: For every real number c, $\lim_{x\to c} x = c$

Theorem 3: **Properties of Limits**
Let *a*, and *c* are real numbers and let n be a positive integer and let *f* and *g* be functions with limits $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then:

1) **Scalar Multiple:** The limit of the product of a constant and a function equals the constant times the limit of the function:
$$\lim_{x\to a}\left[c\cdot f(x)\right] = c\cdot\lim_{x\to a} f(x) = cL$$

2) **Sum or Difference:** The limit of the sum or difference of two functions equals the sum or difference of the limits of the functions:
$$\lim_{x\to a}\left[f(x)\pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$$

3) **Product:** The limit of the product of two functions is the product of the limits of the functions

$$\lim_{x\to a}[f(x)\bullet g(x)]=\left[\lim_{x\to a} f(x)\right]\bullet\left[\lim_{x\to a} g(x)\right]=L\bullet M$$

4) **Quotient:** The limit of a quotient is the quotient of the limits of the numerator and denominator if the limit of the denominator is not zero

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}=\frac{L}{M},\ \text{if}\ \lim_{x\to a} g(x)\neq 0$$

5) **Power:** The limit of a function raised to a power equals the limit of the function raised to the power

   In particular:

$$\lim_{x\to a}\left[\sqrt[n]{f(x)}\right]=\sqrt[n]{\lim_{x\to a} f(x)}=\sqrt[n]{L}$$

Theorem 4: Limits of Polynomial Functions

If $f(x)$ is a polynomial function and *c* is any real number, then $\lim_{x\to c} f(x)=f(c)$.

In other words, the **limit is the value of the polynomial** function *f* at $x=c$.

Theorem 5: Limits of Rational Functions

Let $f(x)$ be a rational function given by $f(x)=\frac{p(x)}{q(x)}$ and let c be a real number such that $q(c)\neq 0$.

Then $\lim_{x\to c} f(x)=f(c)=\frac{p(c)}{q(c)}$.

Theorem 6: Limit Theorem: If *f* and *g* are functions that have limits as *x* approaches *c* and $f(x)=g(x)$ for all $x\neq c$, then $\lim_{x\to c} f(x)=\lim_{x\to c} g(x)$.

Strategy for finding Limits

1) Learn to recognize which limit can be evaluated by direct substitution. (i.e. substitute with the value first)
2) If the limit of $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function g that agrees with f for all x other than x=c.
3) Apply the limit theorem (Theorem 5)

PART 2: **One Sided Limits**

Find: $\lim_{x \to 4} \sqrt{x-4}$

Based on the discussion done in previous sections we can say that $\lim_{x \to 4} \sqrt{x-4} = 0$.

However, $\lim_{x \to 4} \sqrt{x-4}$ = DNE (Does Not Exist). The domain of the expression will not allow the x to approach 4 from the left side, so 4 is approachable from one side only.

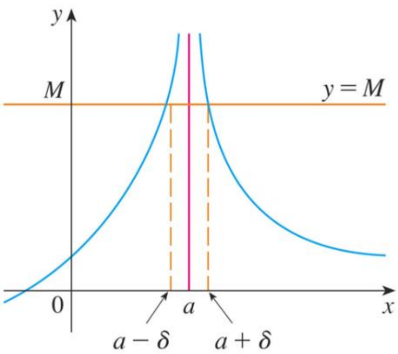
Notation: $x \to a^+$ means x approaches *a* from the right-hand side of a. Hence, $x > a$.
$x \to a^-$ means x approaches *a* from the left-hand side of a. Hence, $x < a$.

Definition 1: (**Right-Hand Limit**) Let f be a function defined on the interval $(a, a + r)$, where $r > 0$. (Note: the function f is defined on the right-hand side of $x = a$.) If as $x \to a^+$, $f(x) \to M$, then we write $\lim_{x \to a^+} f(x) = M$

Definition 2: (**Left-Hand Limit**) Let f be a function defined on the interval $(a - r, a)$, where $r > 0$. (Note: the function f is defined on the left-hand side of $x = a$.) If as $x \to a^-$, $f(x) \to N$, then we write $\lim_{x \to a^-} f(x) = N$

PART 3: **Infinite Limits**

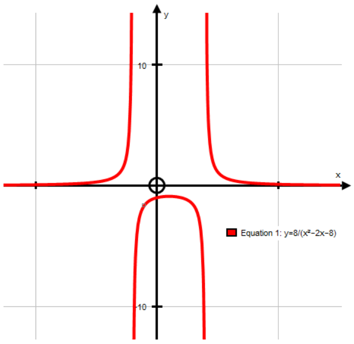
A limit in which $f(x)$ increases or decreases without bound as x approaches a is called an **infinite limit**.



Let us check some examples in which we will study the behavior of the functions at specific values of x.

Therefore, $\lim_{x\to 0} -\frac{5}{x^4} = -\infty$

1) $g(x) = \frac{8}{x^2 - 2x - 8}$ near *x = -2*



The graph shows that the values of $f(x)$ increases *without bound* as *x* approaches -2 from the left and decreases without bound as x approaches -2 from the right.

Therefore, $\lim_{x\to 0} -\frac{5}{x^4} = -\infty$

Infinite Limits and Vertical Asymptotes

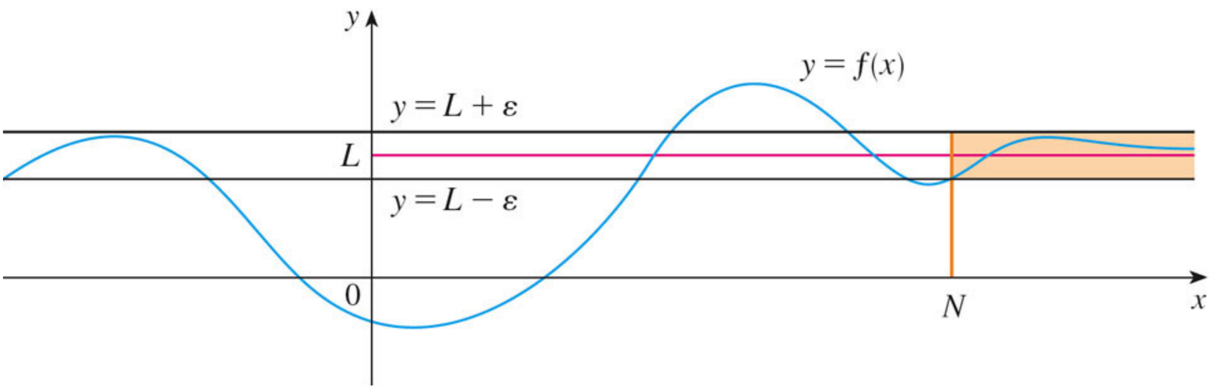
The **vertical line x = c is a vertical asymptote** of the graph of the function f if at least one of the following is true:

$\lim_{x\to c^-} f(x) = \infty$ $\lim_{x\to c^+} f(x) = \infty$ $\lim_{x\to c} f(x) = \infty$

$\lim_{x\to c^-} f(x) = -\infty$ $\lim_{x\to c^+} f(x) = -\infty$ $\lim_{x\to c} f(x) = -\infty$

Limits at Infinity

Let f be a function that is defined for all *x* > *a* for some number *a*. If as *x* takes larger and larger positive values, increasing without bound, the corresponding values of $f(x)$ get very close, and are possibly equal to, a single real number *L* and the values of $f(x)$ can be made arbitrarily close (as close as you want) to *L* by taking large enough values of *x*, then the limit of $f(x)$ as *x* approaches infinity is *L*. $\lim_{x\to\infty} f(x) = L$



Theorem 7: **Limit Theorem:** If c is a constant, then for each positive rational number n,

$\lim_{x\to\infty} \frac{c}{x^n} = 0$ and $\lim_{x\to-\infty} \frac{c}{x^n} = 0$

Limits of Rational Functions: Let p(x), h(x) be polynomials and n is the degree of p(x), m is the degree of h(x):

1) If n = m, then $\lim_{x\to\infty} \frac{p(x)}{h(x)} =$ coefficients of largest power term

2) If n < m, then $\lim_{x\to\infty} \frac{p(x)}{h(x)} = 0$

3) If n > m, then $\lim_{x\to\infty} \frac{p(x)}{h(x)} = \infty$

Limits at infinity and Horizontal Asymptotes

The **horizontal line y=L is a horizontal asymptote** of the graph of the function f if $\lim_{x\to\infty} f(x) = L$