Law of Sines

We saw that the cosines and the sines were useful in solving parts of an oblique triangle if certain other parts are known. The question here is "why are those laws valid?"

An *oblique triangle* is one which contains no right angles. The *angles* from smallest to largest are generally labeled with the uppercase letters as A, B and C and each *side* opposite those angles is labeled with the corresponding lowercase letters a, b and c.

The Law of Sines and the Law of Cosines are usually applied in non-right triangles. The table below describes the cases and the laws that must be used to solve for the other parts of the triangle.

Known Sides & Angles	Description	Use
AAS or ASA	One side and two angles are known	Law of sines
SSS	Three sides are known	Law of cosines
SAS	Two sides and the included angle are known	Law of cosines
SSA	Two sides and the angle opposite one of them are known	Law of sines Ambiguous Case

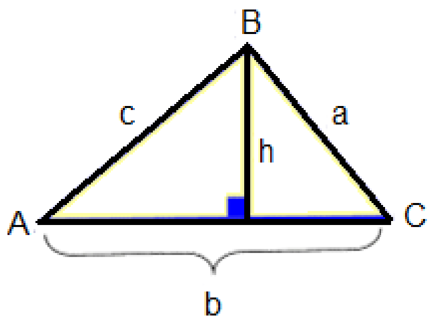
Triangle *ABC* does not contain a right angle. A perpendicular is dropped from vertex *B*. It can now be observed that:

$$\sin A = \frac{h}{c} \Rightarrow h = c\sin A$$

$$\sin C = \frac{h}{a} \Rightarrow h = a\sin C$$

$$h = h \Rightarrow a\sin C = c\sin A$$

$$\therefore \frac{\sin A}{a} = \frac{\sin C}{c}$$



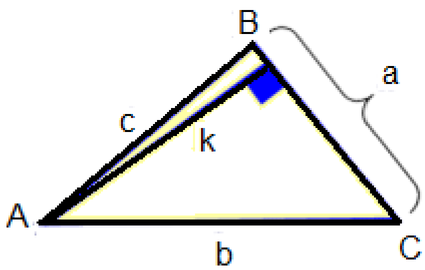
Now, drop a perpendicular from vertex A. It can be observed that:

$$\sin B = \frac{k}{c} \Rightarrow k = c\sin B$$

$$\sin C = \frac{k}{b} \Rightarrow k = b\sin C$$

$$k = k \Rightarrow b\sin C = c\sin B$$

$$\therefore \frac{\sin B}{b} = \frac{\sin C}{c}$$

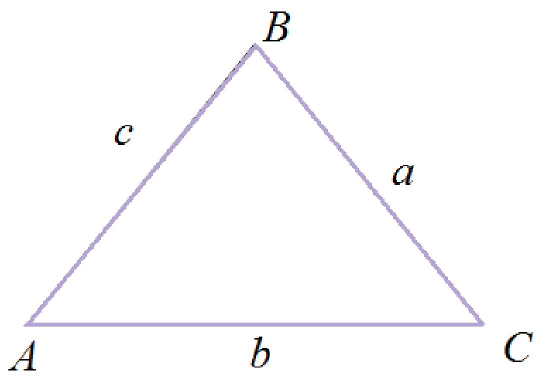


Law 1: <u>Law of Sines:</u> the ratio of the sine of angle A to the length of the side opposite angle A is the same as the ratio of the sine of angle B to the length of the side opposite angle B, and is the same as the ratio of the sine of angle C to the length of the side opposite angle C, for all triangles.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

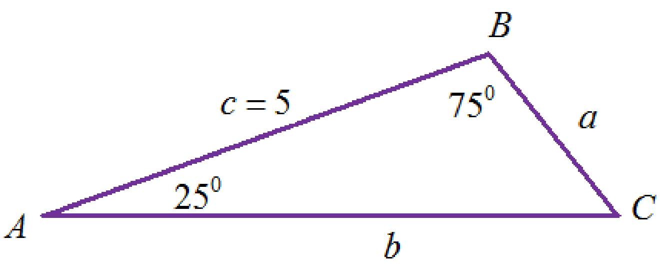
Or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Case 1: **<u>Two Angles and One Side.</u>**
We have one missing angle and two missing sides
$m\angle C = 180^0 - m\angle A - m\angle B$
To find the other sides, we use the Law of Sines.

Example 1: A triangle has two angles of 75° and 25° and the included side is 5. What are the other angle and the two sides?



$m\angle C = 180^0 - m\angle A - m\angle B$

$m\angle C = 180^0 - 25^0 - 75^0 = 80^0$

Using the Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \frac{a}{\sin 25^0} = \frac{b}{\sin 75^0} = \frac{5}{\sin 80^0}$$

By doing a little algebra, we see that $b = \dfrac{5\left(\sin 75^0\right)}{\sin 80^0} = 4.9$

$$a = \frac{5\left(\sin 25^0\right)}{\sin 80^0} = 2.146$$

Case 2: **Two Sides and a *Non*-Included Angle**

When solving for a triangle with 2 known sides and a *non*-included angle, there may be no solution, one solution or two solutions.

In a triangle of sides c, a and its opposite Angle A, the number of solutions is calculated by:

1. If $a < c \bullet \sin A$ no solution
2. If $a \geq c$ one solution.
3. If $c > a > c \bullet \sin A$ two solutions.

Remark: If $a = c \bullet \sin A$ then there is one solution: $m\angle C = 90^0$, $m\angle B = 90^0 - m\angle A$ and $b = c \bullet \cos A$

Three solutions are possible when solving the ambiguous case:

1) NO TRIANGLE:

Example: a = 2, c = 1, $m\angle C = 50^0$

Using the law of sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\Rightarrow \frac{\sin 50^0}{1} = \frac{\sin A}{2}$$

$$\Rightarrow \sin A = 1.5321$$

$\sin A > 1$ Impossible $\Rightarrow$ *There* is no triangle

2) ONE TRIANGLE:

Example: a = 3, b = 2, $m\angle A = 40^0$

Using the law of sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\Rightarrow \frac{\sin 40^0}{3} = \frac{\sin B}{2}$$

$$\Rightarrow \sin B = 0.4285$$

$$\Rightarrow m\angle B_1 = 25.4^0 \text{ or } m\angle B_2 = 154.6^0 \qquad \{\sin \theta = \sin(\pi - \theta)\}$$

Now we need to find $m\angle C$

$$m\angle A + m\angle B + m\angle C = 180^0$$

If $m\angle B_1 = 25.4^0 \Rightarrow 40^0 + 25.4^0 + m\angle C_1 = 180^0 \Rightarrow m\angle C_1 = 114.6^0$

If $m\angle B_2 = 154.6^0 \Rightarrow 40^0 + 154.6^0 + m\angle C_2 = 180^0 \Rightarrow m\angle C_2 = -14.6^0$ Impossible

$\Rightarrow$ *There* is only one triangle

3) TWO TRIANGLES:

Example: a = 6, b = 8, $m\angle A = 35^0$

Using the law of sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\Rightarrow \frac{\sin 35^0}{6} = \frac{\sin B}{8}$$

$$\Rightarrow \sin B = 0.7648$$

$$\Rightarrow m\angle B_1 = 49.9^0 \quad \text{or} \quad m\angle B_2 = 130.1^0 \qquad \{\sin\theta = \sin(\pi - \theta)\}$$

Now we need to find $m\angle C$

$$m\angle A + m\angle B + m\angle C = 180^0$$

$$\text{If } m\angle B_1 = 49.9^0 \Rightarrow 35^0 + 49.9^0 + m\angle C_1 = 180^0 \Rightarrow m\angle C_1 = 95.1^0$$

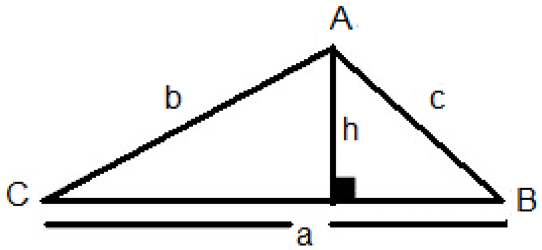
$$\text{If } m\angle B_2 = 130.1^0 \Rightarrow 35^0 + 130.1^0 + m\angle C_2 = 180^0 \Rightarrow m\angle C_2 = 14.9^0$$

$\Rightarrow$ *There* are two triangles

<u>The Area of a Triangle:</u>

We are all familiar with the formula for the area of a triangle, $A = \frac{1}{2}bh$,

where *b* stands for the base and *h* stands for the height drawn to that base.



By using the right triangle on the left side of the diagram, and our knowledge of trigonometry, we can state that: $\sin C = \frac{h}{b} \Rightarrow b\sin C = h$

If we substitute this new expression for the height, we can write the triangle area formula as:

$$Area_{\triangle ABC} = \frac{1}{2}ab\sin C$$

Rule: The area of a triangle equals one-half the product of the lengths of any two sides and the sine of the angle between them.

There are three general formulas to find the area of an oblique triangle.

$$Area_{\triangle ABC} = \frac{1}{2}bc\sin A$$

$$Area_{\triangle ABC} = \frac{1}{2}ac\sin B$$

$$Area_{\triangle ABC} = \frac{1}{2}ab\sin C$$