Law of Cosines

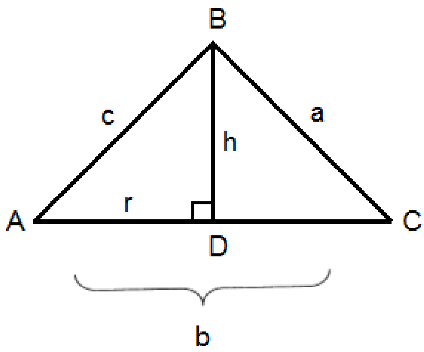
The law of cosines is a formula that relates the three sides of a triangle to the cosines of a given angle. This formula allows us to calculate the side length of non-right triangle as long as you know two sides and an angle and to calculate any angle of a triangle if you know all three side lengths.

The two common problems types that are suitable for the law of cosines are.
Problem type 1: when you know the lengths of 2 sides of a triangle and the angle in between the two sides
Problem type 2: when you know the lengths of three sides of a triangle and want to know a particular angle

Triangle *ABC* at the right does not contain a right angle. A perpendicular is dropped from vertex *B*. It can now be observed that:

$$\sin A = \frac{h}{c} \Rightarrow h = c \bullet \sin A$$

$$\cos A = \frac{r}{c} \Rightarrow r = c \bullet \cos A$$



Using the Pythagorean Theorem in triangle *CBD*, we have: $a^2 = h^2 + (b-r)^2$.
Substituting for *h* and *r* we have:

$$a^2 = (c\sin A)^2 + (b - c\cos A)^2$$

$$\Rightarrow a^2 = c^2\sin^2 A + b^2 - 2bc\cos A + c^2\cos^2 A$$

$$\Rightarrow a^2 = c^2\left(\sin^2 A + \cos^2 A\right) + b^2 - 2bc\cos A \qquad \left\{\sin^2 A + \cos^2 A = 1\right\}$$

$$\Rightarrow a^2 = c^2 + b^2 - 2bc\cos A$$

This same process could be used to produce other lettered statements of this law.

$$b^2 = a^2 + c^2 - 2ac\cos B$$

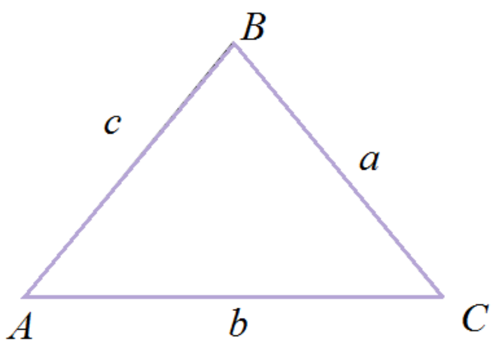
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Law 1: Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Two Sides and the Included Angle

To find the other side we use the Law of Cosines.

$c^2 = a^2 + b^2 - 2ab\cos C$

$b^2 = a^2 + c^2 - 2ac\cos B$

$a^2 = b^2 + c^2 - 2bc\cos A$

It is important to understand that there is only one Law of Cosines even though three "laws" have been listed. All the three are equivalent to each other.

Example 1: A triangle has sides of 4 and 9 with an included angle of 30°. What are the other 2 angles and the length of the other side?

Let's say side a=4 and side b=9 and angle $m\angle C = 30^0$

Using the Law of Cosines:

$c^2 = a^2 + b^2 - 2ab\cos C$

$\Rightarrow c^2 = 4^2 + 9^2 - 2 \bullet 4 \bullet 9\cos 30^0$

$\Rightarrow c^2 = 16 + 81 - 72\left(\frac{\sqrt{3}}{2}\right)$

$\Rightarrow c^2 = 34.6458$

$\Rightarrow c = 5.886$

Using the Law of Sines:

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$\Rightarrow \frac{\sin 30^0}{9} = \frac{\sin A}{4}$

$\Rightarrow \sin A = 0.339$

$\Rightarrow m\angle A = 19.864^0$

Now, we still need to find $m\angle B$:

$m\angle B = 180^0 - m\angle A - m\angle C$

$\Rightarrow m\angle B = 180^0 - 19.864^0 - 30^0$

$\Rightarrow m\angle B = 130.136^0$

All Three Sides Of A Triangle

We use the Law of Cosines when we know all three sides of a triangle. To find the measures of the three missing angles we derive the formulas below from the law of cosines, let us do one derivation:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Rightarrow 2bc\cos A = b^2 + c^2 - a^2$$

$$\Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Therefore,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Rule: Heron's Formula: The area of a triangle with sides a, b, and c is given by

$A = \sqrt{s(s-a)(s-b)(s-c)}$ where s is one-half the perimeter; that is $s = \frac{1}{2}(a+b+c)$.