L'Hopital's Rule

Suppose that *f* and *g* are differentiable functions at *x* = *a* and that is $\lim_{x\to a}\frac{f(x)}{g(x)}$ an indeterminate form of the type $\frac{0}{0}$; that is, $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$. Since *f* and *g* are differentiable functions at *x* = *a*, then *f* and *g* are continuous at *x* = *a*; that is, $f(a)=\lim_{x\to a} f(x)=0$ and $g(a)=\lim_{x\to a} g(x)=0$.

Furthermore, since *f* and *g* are differentiable functions at *x* = *a*, then $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ and $g'(a)=\lim_{x\to a}\frac{g(x)-g(a)}{x-a}$.

Thus, if $g'(a)\neq 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}=\lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}=\frac{f'(a)}{g'(a)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ if f' and g' are continuous at *x* = *a*.

This illustrates a special case of the technique known as **L'Hospital's Rule**.

L'Hospital's Rule for Form $\frac{0}{0}$

L'Hospital's Rule: Suppose that *f* and *g* are differentiable functions on an open interval containing *x* = *a*, except possibly at *x* = *a*, and that $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$. If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ has a finite limit, or if this limit is $(+\infty)$ or $(-\infty)$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$. Moreover, this statement is also true in the case of a limit as $x\to a^-, x\to a^+, x\to -\infty$, or as $x\to +\infty$.

In the following examples, we will use the following three-step process:

Step 1: Check that the limit of $\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$. If it is not, then **L'Hospital's Rule** cannot be used.

Step 2: Differentiate *f* and *g* separately. [*Note*: **Do not differentiate $\frac{f(x)}{g(x)}$ using the quotient rule!**]

Step 3: Find the limit of $\frac{f'(x)}{g'(x)}$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $\frac{f(x)}{g(x)}$. If the limit is an indeterminate form of type $\frac{0}{0}$, then simplify $\frac{f'(x)}{g'(x)}$ algebraically and apply **L'Hospital's Rule** again.

Example 1: Find:

1) $\lim_{x\to 2}\frac{x^2-4}{x-2}$

$\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{0}{0}$

Using L'Hopital's Rule:

$\Rightarrow \lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{2x}{1}=2(2)=4$

2) $\lim_{x\to 0}\frac{\tan 3x}{\sin 2x}$

$\lim_{x\to 0}\frac{\tan 3x}{\sin 2x}=\frac{0}{0}$

Using L'Hopital's Rule:

$\Rightarrow \lim_{x\to 0}\frac{\tan 3x}{\sin 2x}=\lim_{x\to 0}\frac{3\sec^2 3x}{2\cos 2x}=\frac{3(1)}{2(1)}=\frac{3}{2}$

3) $\lim_{h\to 0}\frac{\sqrt[3]{8+h}-2}{h}$

$\lim_{h\to 0}\frac{\sqrt[3]{8+h}-2}{h}=\frac{0}{0}$

Using L'Hopital's Rule:

$\Rightarrow \lim_{h\to 0}\frac{\sqrt[3]{8+h}-2}{h}=\lim_{h\to 0}\frac{\frac{1}{3}(8+h)^{-2/3}(1)}{1}=\lim_{h\to 0}\frac{1}{3(8+h)^{2/3}}=\frac{1}{3(8)^{2/3}}=\frac{1}{12}$

4) $\lim_{x\to \pi/3}\frac{\cos x-1/2}{x-\pi/3}$

$\lim_{x\to \pi/3}\frac{\cos x-1/2}{x-\pi/3}=\frac{0}{0}$

Using L'Hopital's Rule:

$\Rightarrow \lim_{x\to \pi/3}\frac{\cos x-1/2}{x-\pi/3}=\lim_{x\to \pi/3}\frac{-\sin x}{1}=-\sin\left(\pi/3\right)=-\frac{\sqrt{3}}{2}$

5) $\lim\limits_{x\to 0}\dfrac{e^x - x - 1}{x^2}$

$\lim\limits_{x\to 0}\dfrac{e^x - x - 1}{x^2} = \dfrac{0}{0}$

Using L'Hopital's Rule:

$\Rightarrow \lim\limits_{x\to 0}\dfrac{e^x - x - 1}{x^2} = \lim\limits_{x\to 0}\dfrac{e^x - 1}{2x} = \dfrac{0}{0}$

Using L'Hopital's Rule:

$\Rightarrow \lim\limits_{x\to 0}\dfrac{e^x - x - 1}{x^2} = \lim\limits_{x\to 0}\dfrac{e^x - 1}{2x} = \dfrac{0}{0}$

Using L'Hopital's Rule:

$\lim\limits_{x\to 0}\dfrac{e^x - 1}{2x} = \lim\limits_{x\to 0}\dfrac{e^x}{2} = \dfrac{1}{2}$

6) $\lim\limits_{x\to +\infty}\dfrac{1/x^2}{\sin\left(1/x\right)}$

$\lim\limits_{x\to +\infty}\dfrac{1/x^2}{\sin\left(1/x\right)} = \dfrac{0}{0}$

Using L'Hopital's Rule:

$\Rightarrow \lim\limits_{x\to +\infty}\dfrac{1/x^2}{\sin\left(1/x\right)} = \lim\limits_{x\to +\infty}\dfrac{-2/x^3}{\cos\left(1/x\right)\left(-1/x^2\right)} = \lim\limits_{x\to +\infty}\dfrac{2/x}{\cos\left(1/x\right)} = \dfrac{0}{1} = 0$

L'Hospital's Rule for Form $\dfrac{\infty}{\infty}$

Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = 0$. If $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ has a finite limit, or if this limit is $(+\infty)$ or $(-\infty)$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$. Moreover, this statement is also true in the case of a limit as $x \to a^-, x \to a^+, x \to -\infty$, or as $x \to +\infty$.

Indeterminate Form of the Type $0 \cdot \infty$

Indeterminate forms of the type $0 \cdot \infty$ can sometimes be evaluated by rewriting the product as a quotient, and then applying **L'Hospital's Rule** for the indeterminate forms of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Indeterminate Form of the Type $\infty - \infty$

A limit problem that leads to one of the expressions $(+\infty)-(+\infty)$, $(-\infty)-(-\infty)$, $(+\infty)+(-\infty)$, $(-\infty)+(+\infty)$ is called an **indeterminate form of type** $\infty-\infty$. Such limits are indeterminate because the two terms exert conflicting influences on the expression; one pushes it in the positive direction and the other pushes it in the negative direction. However, limits problems that lead to one the expressions $(+\infty)+(+\infty)$, $(+\infty)-(-\infty)$, $(-\infty)+(-\infty)$, $(-\infty)-(+\infty)$ are not indeterminate, since the two terms work together (the first two produce a limit of $+\infty$ and the last two produce a limit of $-\infty$). Indeterminate forms of the type $\infty-\infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Indeterminate Forms of the Type $0^0, \infty^0, 1^\infty$

Limits of the form $\lim_{x\to a}[f(x)]^{g(x)}$ $\left\{or \lim_{x\to\infty}[f(x)]^{g(x)}\right\}$ frequently give rise to indeterminate forms of the types $0^0, \infty^0, 1^\infty$. These indeterminate forms can sometimes be evaluated as follows:

$$y=[f(x)]^{g(x)}$$
$$\ln y=\ln[f(x)]^{g(x)}=g(x)\ln[f(x)]$$
$$\lim_{x\to a}[\ln y]=\lim_{x\to a}\{g(x)\ln[f(x)]\}$$

The limit on the right hand side of the equation will usually be an indeterminate limit of the type $0\cdot\infty$. Evaluate this limit using the technique previously described. Assume that $\lim_{x\to a}\{g(x)\ln[f(x)]\}=L$.

Finally, $\lim_{x\to a}[\ln y]=L \Rightarrow \ln\left[\lim_{x\to a} y\right]=L \Rightarrow \lim_{x\to a} y=e^L$.