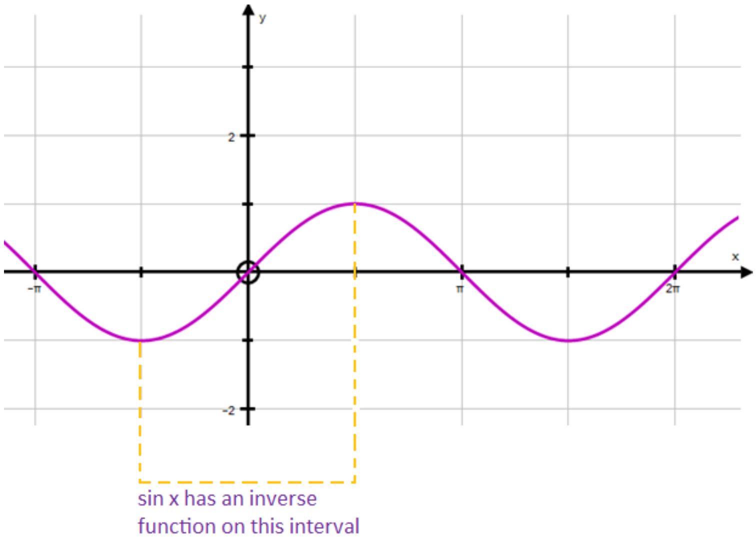
Inverse Trigonometric Functions

From Pre – Calculus, for a function to have an inverse it must be one – to – one, that is it must pass the Horizontal Line Test. The function y=sin(x) does not pass the test because different values of x yield the same y-value.



However, if you restrict the domain to the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, the following properties holds:

1) On the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the function $y = \sin x$ is increasing.
2) On the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ takes the full range of values, $-1 \le \sin x \le 1$
3) On the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y = \sin x$ is one – to – one.

So, on the restricted domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $y = \sin x$ has a unique inverse function called the **inverse sine function**, and is denoted by: $y = \arcsin x$ or $y = \sin^{-1} x$.

The notation $\sin^{-1} x$ is consistent with the inverse function notation $f^{-1}(x)$. The "$\arcsin x$"comes from the association of a central angle with intercepted arc length on a unit circle. So, "$\arcsin x$ "means the angle (or arc) whose sine is x.

Remark: $\sin^{-1} x$ denotes the inverse of the sine function rather than $\frac{1}{\sin x}$.

Definition 1: The inverse sine function is defined by: $y = \arcsin x$ if and only if $\sin y = x$

Where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. The domain of $y = \arcsin x$ is $[-1,1]$, and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Example 1: If possible, find the exact value:

1) $\arcsin\left(-\frac{1}{2}\right)$

   $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ it follows that $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

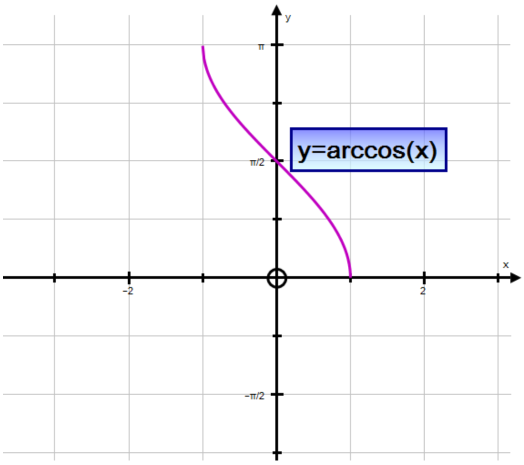
2) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

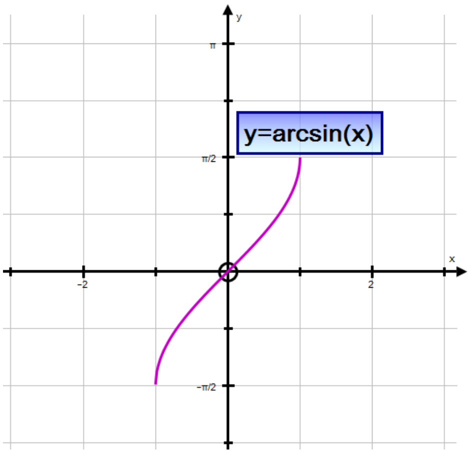
   $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ it follows that $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

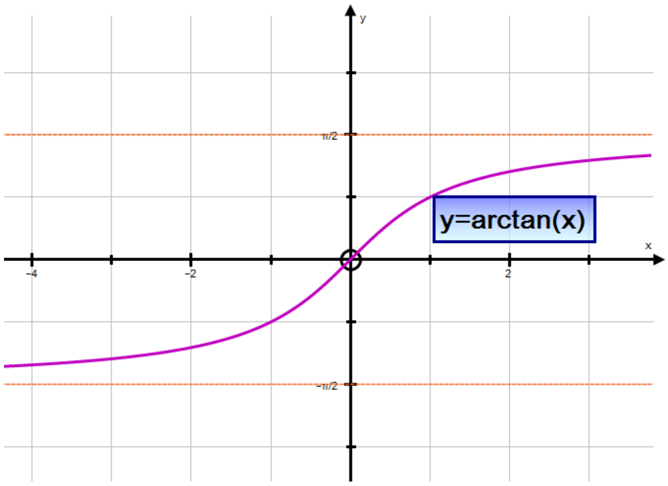
Definition 2: The inverse trigonometric functions:

Function	Domain	Range
$y = \arcsin x \Leftrightarrow \sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x \Leftrightarrow \cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x \Leftrightarrow \tan y = x$	$-\infty \le x \le \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

The graphs of these three inverse trigonometric functions are:







Calculators and inverse functions

We can evaluate the inverse functions using calculators. To evaluate $\arcsin\left(\frac{1}{2}\right)$ we do the following:

shift	sin	$\frac{1}{2}$

Composite of functions:
For all x in the domain of f and f^{-1}, the inverse functions have the properties:
$f\left(f^{-1}(x)\right)=x$ and $f^{-1}\left(f(x)\right)=x$

Properties: Inverse properties of trigonometric functions:

1. If $-1\le x\le 1$ and $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$, then:

   $\sin(\arcsin x)=x$ and $\arcsin(\sin y)=y$

2. If $-1\le x\le 1$ and $0\le y\le\pi$, then:

   $\cos(\arccos x)=x$ and $\arccos(\cos y)=y$

3. If x is any real number and $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$, then:

   $\tan(\arctan x)=x$ and $\arctan(\tan y)=y$