Inverse Functions

Consider the function $f(x)=5x+2$. What does this function do to the input number *x*?
First, it multiplies the input number by 5, and then it adds 2 to the resulting product.
Now, suppose you were told that 27 was the resulting output number. What would you do to find out what the input number was? First, you would have to *subtract* 2 then *divide* by 5 (getting 5). That is, you would have to undo the operations that *f* did, and in the reverse order. This function that subtracts two then divides by five is called the *inverse* of *f*, and is written $f^{-1}(x)$. It is the function that *undoes* what *f does*. So if $f(x)=5x+2$, then $f^{-1}(x)=\frac{x-2}{5}$

Now consider the function $g(x)=x^2$, the squaring function. How would one undo the squaring function? By taking the square root? What if the input number is - 3? Then *g* (- 3) = 9. But if we take the square root of 9, we get 3, not - 3. The problem is that there are two input numbers, - 3 and + 3, with the same output number 9. Thus, there is no way to figure out which input number, - 3 or + 3, produced the 9, so there is no way to undo what *g* did. Remember, a function must always give the same output for any given input. It cannot sometimes give an output of 3 when the input is 9 and other times give an output of - 3 when the input is 9. Thus there is no inverse function for *g*. This dilemma will always occur for functions that transform two or more input numbers into the *same* output number.

In order to have an inverse, a function must have only one *input* number corresponding to any given *output* number.

We already know, from the definition of a function, that there can be only one output number for any given input number. If we add the requirement that there must *also* be only one input number for any given output number, we get what is called a *one-to-one function*.
Only one-to-one functions have an inverse function.

Definition 1: A function is a one-to-one function if and only if each second element corresponds to one and only one first element. (Each x and y value is used only once).

A function f is one-to-one if, for every a and b in its domain, $f(a)=f(b)\Leftrightarrow a=b$

Definition 2: Functions f(x) and g(x) are inverses of each other if and only if
$(f\circ g)(x)=(g\circ f)(x)=x$

Steps for finding the inverse of a function $f(x)$

Step 1: Replace f(x) by y in the equation describing the function.
Step 2: Check if the function is one-to-one function.
Step 3: Solve for x.
Step 4: Interchange x and y. In other words, replace every x by y and vice versa.
Step 5: Replace y by $f^{-1}(x)$.

Example 1: Let function f be defined by the set of ordered pairs $f(x)$ = {(1, 0), (4, 5), (6, 9)}
The function is a one-to-one function. By interchanging the first and second coordinates of each ordered pair in f, we can define the inverse function $f^{-1}(x)$ = {(0, 1), (5, 4), (9, 6)}

Definition 3: Let $f^{-1}(x)$ be the inverse of $f(x)$

- The domain of $f(x)$ is equal to the range of $f^{-1}(x)$
- The range of $f(x)$ is equal to the domain of $f^{-1}(x)$

Example 2: Let $f(x)$ = x + 2, and $g(x)$ = x − 2. Prove that: $(f \circ g)(x) = (g \circ f)(x) = x$

$$(f \circ g)(x) = f(x-2) = (x-2)+2 = x$$

$$(g \circ f)(x) = g(f(x)) = g(x+2) = (x+2)-2 = x$$

Horizontal line test: If it is possible for a horizontal line to intersect the graph of a function more than once, then the function is **not** one-to-one and its inverse is not a function.

The following three statements are equivalent:

- The function f has an inverse
- The graph of f is cut at most once by any horizontal line
- The function does not have the same value at two distinct points in its domain.

Example 3: If the function is one-to-one, find a formula for the inverse:

$$y = \frac{5x-3}{2x+1}$$

$$x = \frac{5y-3}{2y+1}$$

$$x(2y+1) = 5y-3$$
$$2xy + x = 5y-3$$
$$2xy - 5y = -x-3$$
$$y(2x-5) = -x-3$$
$$y = -\frac{x+3}{2x-5}$$

$$f^{-1}(x) = -\frac{x+3}{2x-5}$$

Rule 1: The graph of f^{-1} is a reflection of the graph of f across the line $y = x$.

Example 4: Find the inverse of the function $f(x) = \frac{2}{x-1} \quad x \in R,\ x > 1.$

Check if $f(x)$ *is one to one*

$$f(a) = f(b)$$
$$\frac{2}{a-1} = \frac{2}{b-1}$$
$$2(a-1) = 2(b-1)$$
$$a-1 = b-1$$
$$a = b$$

Therefore, $f(x)$ *is one to one*

$$y = f(x) = \frac{2}{x-1}$$

$\Rightarrow$xy – y = 2

$\Rightarrow$xy = 2 + y

$\Rightarrow$ x = $\frac{2}{y} + 1$

So the inverse function is $f^{-1}(x) = 1 + \frac{2}{x}$

If you draw the graph of y = f(x), y = f^{-1}(x) and y = x on the same axes, the function and its inverse are symmetric with respect to the line y = x.

Theorem 1: Inverse Functions and Composition

If a function f is one-to-one, then f^{-1} is the unique function such that each of the following holds:

$$\left(f^{-1} \circ f\right)(x) = f^{-1}\left(f(x)\right) = x, \quad \text{for each } x \text{ in the domain of } f, \text{ and}$$
$$\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = x, \quad \text{for each } x \text{ in the domain of } f^{-1}.$$

Example 5: Show that $f(x) = \frac{2}{x-1}$ and $g(x) = 1 + \frac{2}{x}$ are inverses of each other

$$(g \circ f)(x) = g\left(f(x)\right) = g\left(\frac{2}{x-1}\right) = 1 + \frac{2}{\frac{2}{x-1}} = 1 + x - 1 = x$$

$$(f \circ g)(x) = f\left(g(x)\right) = f\left(1 + \frac{2}{x}\right) = \frac{2}{\left(1 + \frac{2}{x}\right) - 1} = \frac{2}{1 + \frac{2}{x} - 1} = \frac{2}{\frac{2}{x}} = x$$

$$(g \circ f)(x) = (f \circ g)(x) = x$$

Therefore, $f(x) = \frac{2}{x-1}$ and $g(x) = 1 + \frac{2}{x}$ are inverses of each other