Integration by Substitution

For differentiation we had special rules like chain rule, product rule, and quotient rule. Since integration is the inverse process of differentiation, it seems reasonable to conclude that special rules exist for integration. One special rule we will study for integration is called **substitution,** commonly called u-substitution. Integration by substitution allows us to recover functions that required the chain rule to take its derivative.

The basic strategy of u-substitution is to find one part of the integral that 'looks like' the derivative of another part. We then substitute the variable *u* for one part, and *du* for the other (the derivative part). There are usually 'tipoff's' that we would like to *u-sub*. One in particular is the presence of two polynomials within the integrand, which differ in degree by 1.

For example, to find the derivative of $f(x)=(x^2+1)^3$, we use the chain rule and set $u=x^2+1$, so $u'=2x$.

Thus, when given $f(x)=(x^2+1)^3=u^3$

$$\begin{aligned} f'(x) &= 3u^2\cdot u' \\ &= 3(x^2+1)^2\cdot(2x) \\ &= 6x(x^2+1)^2 \end{aligned}$$

Thus, $\frac{d}{dx}(x^2+1)^3=6x(x^2+1)^2$ or equivalently $\int 6x(x^2+1)^2dx=(x^2+1)^3+C$.

But, how can we recover $f(x)$ given $f'(x)=6x(x^2+1)^2$? We must use substitution. We begin by letting $u=x^2+1$ and then take the derivative of both sides with respect to *x*

$\frac{du}{dx}=2x$ and solve the equation for *dx*: $\frac{du}{2x}=dx$

Now substitute *u* for x^2+1 and $\frac{du}{2x}$ for *dx*, then simplify.

$$\begin{aligned} \int 6x(x^2+1)^2\,dx &= \int 6x\,u^2\frac{du}{2x} \\ &= \int 3u^2\,du \\ &= 3\int u^2du \end{aligned}$$

We can use the general power rule for antiderivatives on the last integral:

$$3\left(\frac{1}{3}u^3\right)+C=u^3+C$$

Finally we must substitute x^2+1 for *u* and we have recovered $f(x)$ from above.

$(x^2+1)^3+C$.

Rule 1: It might be difficult to determine what expression to make *u*. The table below shows different types of problems that involve substitution. In each case let u = *f(x)*.

$\int\left[f'(x)\cdot(f(x))^n\right]dx$	$\int\left[f'(x)\cdot e^{f(x)}\right]dx$	$\int\left[f'(x)\cdot\sqrt[n]{f(x)}\,\right]dx$	$\int\frac{f'(x)}{f(x)}\,dx$
$\int\left[f'(x)\bullet\sin(f(x))\right]dx$			

Example 1: Find:

1) $\int x^2\sqrt{x^3+1}\,dx$

Since $f(x)=x^3+1$ we let $u=x^3+1$. Now differentiate this equation with respect to *x* and then solve for *dx*.

$$\frac{du}{dx}=3x^2 \quad\Rightarrow\quad du=3x^2dx \quad\Rightarrow\quad \frac{du}{3x^2}=dx$$

Now substitute *u* for x^3+1 and $\frac{du}{3x^2}$ for *dx*.

$$\int x^2\sqrt{x^3+1}\;dx=\int x^2\sqrt{u}\;\frac{du}{3x^2}$$

After simplifying the integral will be completely in terms of *u*.

$$=\int\frac{1}{3}\sqrt{u}\;\;du$$

$$=\frac{1}{3}\int(u)^{1/2}\;\;du$$

$$=\frac{1}{3}\left(\frac{2}{3}u^{3/2}\right)+C$$

$$=\frac{2}{9}u^{3/2}+C$$

Now we must substitute x^3+1 for *u* to obtain the final answer.

$$\frac{2}{9}\left(x^3+1\right)^{3/2}+C$$

2) $\int 4xe^{2x^2+1}\;\;dx$

Since $f(x)=2x^2+1$ we let $u=2x^2+1$ and then differentiate both sides of the equation with respect to *x*

$$\frac{du}{dx}=4x \quad\Rightarrow\quad du=4x\,dx \quad\Rightarrow\quad \frac{du}{4x}=dx\,.$$

Now substitute *u* for $2x^2+1$, $\frac{du}{4x}$ for *dx*, and the new limits of integration.

$$\int 4xe^{2x^2+1}\;\;dx=\int 4x\,e^u\,\frac{du}{4x}=\int e^u du=e^u+C=e^{2x^2+1}+C$$

3) $\int \left(\frac{3x^2+2}{x^3+2x} \right) dx$

Let $u = x^3 + 2x$, then

$$\frac{du}{dx} = 3x^2 + 2 \quad \Rightarrow \quad du = \left(3x^2+2\right)dx \quad \Rightarrow \quad \frac{du}{3x^2+2} = dx$$

Making all substitutions we get the following integral.

$$\int \frac{3x^2+2}{u} \cdot \frac{du}{3x^2+2} = \int \frac{du}{u} = \ln|u| + C = \ln\left|x^3+2x\right| + C$$