Integration by Parts

By the **Product Rule for Derivatives,** $\frac{d}{dx}\{f(x)g(x)\} = f(x)g'(x) + g(x)f'(x)$. Thus,

$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x) \Rightarrow \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx =$

$f(x)g(x) \Rightarrow \int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$.

This formula for integration by parts often makes it possible to reduce a complicated integral involving a product to a simpler integral.

By letting $u = f(x) \Rightarrow du = f'(x)dx$

$dv = g'(x)dx \Rightarrow v = g(x)$

We get the more common formula for integration by parts: $\int udv = uv - \int vdu$

The main steps of this technique are:

Step 1: Assign variables

The problems that most students encounter with this formula are the substitutions for u, dv, v, and du. Two possibilities exist for substitutions.

a) If one of the functions cannot be integrated, assign u to the function.

b) If both functions can be integrated, then assign u to the function that eventually differentiates to zero.

Step 2: Integrate and differentiate correct functions

Once u and dv have been assigned their proper functions, integration and differentiation can begin. Setting up two columns to do this is very helpful. The function assigned to **u** will be **differentiated**, and the function assigned to **dv** will be **integrated**.

Step 3: Apply integration by parts formula

Applying the formula takes only two basic steps:

1. Substitute u, v, du, and dv into the formula $\int udv = uv - \int vdu$
2. Calculate uv - $\int vdu$

If $\int vdu$ is difficult or impossible to integrate, go back to step 1 and consider other choices for u and dv.

Step 4: Repeat if necessary

Example 1: Find. $\int x\ln x dx$

Let $u=\ln x \Rightarrow du=\frac{1}{x}dx$

$dv = xdx \Rightarrow v=\int x\,dx=\frac{1}{2}x^2$.

Thus,

$\int x\ln x dx=\int(\ln x)(x\,dx)=\int udv$

$=uv-\int vdu$

$=(\ln x)\left(\frac{1}{2}x^2\right)-\int\left(\frac{1}{2}x^2\right)\left(\frac{1}{x}dx\right)$

$=\frac{1}{2}x^2\ln x-\frac{1}{2}\int x\,dx$

$=\frac{1}{2}x^2\ln x-\frac{1}{2}\left(\frac{1}{2}x^2\right)+C$

$=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$

Strategy for Integration by Parts

1. $\int x^n f(x)dx$ where $f(x)=\sin ax, \cos ax, e^{ax}$ (where $a>0$)
   Let $u=x^n$ and $dv=f(x)dx$ (use tabular method when $n \geq 2$)

2. $\int x^n f(x)dx$ where $f(x)=\ln ax, \sin^{-1}ax, \tan^{-1}ax, \sec^{-1}ax$ (where $a>0$)
   Let $u=f(x)$ and $dv=x^n dx$ (including the case n = 0)

3. $\int e^{ax}\sin(bx)dx$ and $\int e^{ax}\cos(bx)dx$ (where a and b nonzero constants)

   You have to use integration by parts **twice**.

   The first time udv-substitution is arbitrary, for example, $\int e^{ax}\sin(bx)dx$, you can let $u=e^{ax}$ and $dv=\sin(bx)dx$ or $u=\sin(bx)$ and $dv=e^{ax}dx$. However, the second time udv -substitution must follow the same substitution you use for the first time. Of course, the most important thing is, after you carry out the integration by parts twice, you will see the integral $\int e^{ax}\sin(bx)dx$ again (but with minus sign before it). What you need to do now is to transpose this negative integral to the other side.

Tabular Integration

A problem arises when we attempt an example like:
$\int x^7 e^x dx$ *or* $\int x^7 \cos(x) dx$

For problems like these, where more than two substitutions into $\int u dv = uv - \int v du$ are required, an alternative form to integration by parts is recommended. This "book-keeping" method allows one to do integration by parts without having to make all the tedious substitutions.

Step 1: Identify the term "*u*" that will yield zero after repeated differentiation.

Step 2: Identify the term "*v*" that can be easily integrated.

Step 3: Make two columns; label the right column *vdv* and the left column *u*.

Step 4: Differentiate "*u*" repeatedly until it becomes zero, and make sure to alternate between "-" and "+". Always start off with a "-".

Step 5: Integrate the "*v*" term as many times as you differentiated the "*u*" term.

Step 6: Multiply the "*u*" term by the "*v*" term just below its row.

$u = x^7$	$vdv = e^x dx$
$(-)7x^6$	e^x
$(+)42x^5$	e^x
$(-)210x^4$	e^x
$(+)840x^3$	e^x
$(-)2520x^2$	e^x
$(+)5040x$	e^x
$(-)5040$	e^x
$(+)0$	e^x

The result is:
$\int x^7 e^x dx = x^7 e^x - 7x^6 e^x + 42x^5 e^x - 210x^4 e^x + 840x^3 e^x - 2520x^2 e^x + 5040xe^x - 5040e^x + c$

Integrals of the form $\int f(x)g(x)\,dx$, in which **f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty**, can be evaluated using tabular integration.

Reduction Formulas

Integration by parts can be used to derive **reduction formulas** for integrals. These are formulas that express an integral involving a power of a function in terms of an integral that involves a *lower* power of that function