Half Angles and Power Reducing Formulas

Half Angle Formula - Sine

Now, if we let $\theta = \frac{\alpha}{2}$ then 2θ = α and our formula becomes: $\cos\alpha = 1 - 2\sin^2\left(\frac{\alpha}{2}\right)$

We now solve for $\sin\left(\frac{\alpha}{2}\right)$ (that is, we get $\sin\left(\frac{\alpha}{2}\right)$ on the left of the equation and everything else on the right):

$$2\sin^2\left(\frac{\alpha}{2}\right) = 1 - \cos\alpha$$

$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2} \Rightarrow \sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Rule 1: **sine of a half-angle** identity:

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

The sign of $\sin\frac{\alpha}{2}$ depends on the quadrant in which $\frac{\alpha}{2}$ lies.

- If $\frac{\alpha}{2}$ is in the **first or second quadrants**, the formula uses the positive case:

  $$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos\alpha}{2}}$$

- If $\frac{\alpha}{2}$ is in the **third or fourth quadrants**, the formula uses the negative case:

  $$\sin\left(\frac{\alpha}{2}\right) = -\sqrt{\frac{1-\cos\alpha}{2}}$$

Half Angle Formula - Cosine

Using a similar process, with the same substitution of $\theta = \frac{\alpha}{2}$ (so 2θ = α) we substitute into the identity: $\cos 2\theta = 2\cos^2\theta - 1$

We obtain:

$$\cos 2\left(\frac{\alpha}{2}\right) = 2\cos^2\frac{\alpha}{2} - 1$$

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1$$

Reverse the equation: $2\cos^2\frac{\alpha}{2} - 1 = \cos\alpha$

Add 1 to both sides: $2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha$

Divide both sides by 2: $\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$

Solving for $\cos(\frac{\alpha}{2})$, we obtain: $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

Rule 2: **Cosine of a half-angle** identity:

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

As before, the sign we need depends on the quadrant.

- If $\frac{\alpha}{2}$ is in the **first or fourth quadrants**, the formula uses the positive case:
  $\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$
- If $\frac{\alpha}{2}$ is in the **second or third quadrants**, the formula uses the negative case:
  $\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}$

Example 1: Let: $\alpha = \frac{\pi}{2}$ and $\beta = \beta$, verify that $\cos\left(\frac{\pi}{2} + \beta\right) = -\sin\beta$

Find the value of cos 15^0, using the ratios of 30^0only.

$$\frac{\sqrt{3}}{2} = 2\cos^2 15^0 - 1$$

$$\cos^2 15^0 = \frac{\frac{\sqrt{3}}{2} + 1}{2}$$

$$\cos^2 15^0 = \frac{\sqrt{3}+2}{4}$$

$\cos 15^0 = \pm\sqrt{\frac{\sqrt{3}+2}{4}}$ But 15^0 is in Quadrant I

$$\Rightarrow \cos 15^0 = +\sqrt{\frac{\sqrt{3}+2}{4}}$$

$$\Rightarrow \cos 15^0 = +\frac{\sqrt{\sqrt{3}+2}}{2}$$

Rule 3: The following identities are true for all values for which they are defined and they are called power reducing formulas:

- $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$
- $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$
- $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$

$\cot^2\theta = \dfrac{1+\cos 2\theta}{1-\cos 2\theta}$