Geometric Sequence and Series

A geometric sequence is a sequence such that each successive term is obtained from the previous term by multiplying by a fixed number called a ratio.
The sequence 5, 10, 20, 40, 80 ... is an example of a geometric sequence. The pattern is obtained by multiplying by a fixed number 2 to get to the next term.
Be careful that you don't think that every sequence that has a pattern in multiplication is geometric. It is geometric if you are always multiplying by the SAME number each time.

Definition: A *geometric series* (or *geometric progression*) is a sequence of numbers such that each number bears a constant ratio, called the *common ratio*, to the previous number.
If a_1 is the first term, a_n is the n^{th} term, r is the common ratio, n is the number of terms then:

$$a_n = a_1 r^{n-1}$$

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots = \frac{a_n}{a_{n-1}}$$

Example 1: Find the first five terms and the common ratio of the geometric sequence $a_n = 5(3)^n$.

n is our term number and we plug the term number into the function to find the value of the term. Let's see what we get for our first five terms:

$$a_n = 5(3)^n$$

$$a_1 = 5(3)^1 = 15$$

$$a_2 = 5(3)^2 = 45$$

$$a_3 = 5(3)^3 = 5(27) = 135$$

$$a_4 = 5(3)^4 = 5(81) = 405$$

$$a_5 = 5(3)^5 = 5(243) = 1215$$

What would be the common ratio for this problem?

$$r = \frac{a_2}{a_1} = \frac{45}{15} = 3$$

Example 2: Find the common ratio for a geometric sequence with a first term of $\frac{3}{4}$ and a third term of $\frac{27}{16}$.

We will use the n^{th} term formula for a geometric sequence, $a_n = a_1 r^{n-1}$, to help us with this problem. We will just be looking for *r*.
Plugging in $\frac{3}{4}$ for a_1, 3 for *n*, and $\frac{27}{16}$ for the n^{th} term we get:

$a_n = a_1 r^{n-1}$

$\Rightarrow a_3 = a_1 r^{3-1}$

$\Rightarrow \frac{27}{16} = \frac{3}{4} r^2$

$\Rightarrow r^2 = \left(\frac{4}{3}\right)\left(\frac{27}{16}\right) = \frac{9}{4}$

$\Rightarrow r = \pm\sqrt{\frac{9}{4}}$

$\Rightarrow r = \pm\frac{3}{2}$

The common ratio could be either $-\frac{3}{2}$ or $+\frac{3}{2}$

Example 3: Write a formula for the n^{th} term of the geometric sequence 16, - 4, 1, -1/4 ...
The n^{th} term formula for a geometric sequence is $a_n = a_1 r^{n-1}$

The first term of this sequence is 16.

$r = \frac{a_2}{a_1} = \frac{-4}{16} = -\frac{1}{4}$

Putting in 16 for a_1 and $-\frac{1}{4}$ for *r* we get:

$a_n = a_1 r^{n-1}$

$\Rightarrow a_n = 16\left(-\frac{1}{4}\right)^{n-1}$

Example 4: Calculate a_{100} for the geometric sequence with firs term a_1 = 35 and common ratio r = 1.05.

Since you know a_1, r, and n, you can multiply the first term by (100 – 1) common ratios.

$a_{100} = 35 \times 1.05^{100-1}$

$= 35 \times 1.05^{99}$

≈ 4383.375262

Example 5: A geometric sequence has a_1 = 17 and r = 2. If a_n = 34816, find n.

Substituting in the formula gives

$a_n = a_1 \cdot r^{n-1}$

$34816 = 17 \times 2^{n-1}$

$2048 = 2^{n-1}$

Taking the log of each member,

Log 2048 = log 2 $^{n-1}$

Log 2048 = (n – 1) log 2 Log of a power

$$\frac{\log 2048}{\log} = n-1$$

11 = n -1

12 = n

Rule 1: If we have the n^{th} term and the m^{th} term of a geometric sequence then the common ratio can be calculated using the rule

$$a_n = a_m r^{n-m}$$

Rule 2: the sum of terms in a geometric sequence is:

$$S_n = a_1 \frac{(1-r^n)}{(1-r)}$$

In this formula:

S_n is the sum of the first n terms in a sequence

a_1 is the first term in the sequence

r is the common ratio in the geometric sequence

n is the number of terms you are adding up

Example 6: Find the sum of the finite geometric series 3 - 6 + 12 - 24 + 48 - 96.

The sum of the first *n* terms of geometric sequence, $S_n = a_1 \frac{(1-r^n)}{(1-r)}$

The first term is $a_1 = 3$

What is *r*, the common ratio?

Note that you would have to multiply -2 each time you go from one term to the next: (3)(-2) = - 6, (-6)(2) = -12, (-12)(-2) = 24, (24)(-2) = - 48, and (-48)(-4) = -96.

How many terms are we summing up?

The number of terms we are summing up is 6.

Putting in 3 for the first term, -2 for the common ratio, and 6 for *n*, we get:

$$S_n = a_1 \frac{(1-r^n)}{(1-r)}$$

$$S_n = 3 \times \frac{(1-(-2)^6)}{(1-(-2))} = \frac{3(1-64)}{3} = -63$$

Example 7: Find the sum of the finite geometric series $\sum_{i=0}^{20}\left[3(1.1)^{i}\right]$.

What is a_1, the first term? 3
Since this summation starts at 0, you need to plug in 0 into the given formula:
$3(1.1)^{0} = 3(1) = 3$

What is *r*, the common ratio?
Note that 1.1 is the number that is being raised to the exponent. So each time the number goes up on the exponent, in essence you are multiplying it by 1.1

How many terms are we summing up?
If you start at 0 and go all the way to 20, there will be 21 terms.

Putting in 3 for the first term, 1.1 for the common ratio, and 21 for *n*, we get:

$$S_n = a_1 \frac{(1-r^n)}{(1-r)}$$

$$\sum_{i=0}^{20}\left[3(1.1)^{i}\right] = 3\times\frac{(1-(1.1)^{21})}{(1-1.1)} \approx 192.007$$

Rule 3: The Sum of an Infinite Geometric Series

- If -1 < *r* < 1 (or $|r|<1$), then the sum of the infinite geometric series

  $a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots$ in which a_1 is the first term and *r* is the common ratio is given by $S = \frac{a_1}{(1-r)}$.

- If $|r| \geq 1$, the sum of the infinite series cannot be calculated.

Example 8: Find the sum of the infinite series $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots$, if possible.

We will use the formula for the sum of infinite geometric sequence, $S = \frac{a_1}{(1-r)}$, to help us with this problem.
Basically we need to find two things: the first term of the sequence and the common ratio.

What is the first term, a_1? 2
What is the common ratio, *r*?
Note that you would have to multiply 1/3 each time you go from one term to the next: (2)(1/3) = 2/3, (2/3)(1/3) = 2/9, (2/9)(1/3) = 2/27. It has to be consistent throughout the sequence.

Putting in 2 for the first term and 1/3 for the common ratio we get:

$$S = \frac{a_1}{(1-r)}$$

$$S = \frac{2}{\left(1-\frac{1}{3}\right)} = \frac{2}{\frac{2}{3}} = 3$$

Example 9: Find the sum of the infinite series $1.5+3+6+12+\ldots$, if possible.

The first term, a_1 is 1.5

What is the common ratio, *r*?
Note that you would have to multiply 2 each time you go from one term to the next: (1.5)(2) = 3, (3)(2) = 6, (6)(2) = 12. It has to be consistent throughout the sequence.

Since the geometric ratio is 2 and $|2| \geq 1$, the sum cannot be determined.

Example 10: Find the sum of the infinite series $\sum_{n=0}^{\infty}\left[(-5)(-0.5)^n\right]$, if possible.

The first term a_1 is -5
Since this summation starts at 0, you need to plug in 0 into the given formula:

$$(-5)(-0.5)^n \ldots\ldots\ldots\ldots \text{ for } n=0$$

$$(-5)(-0.5)^0 = -5(1) = -5$$

What is the common ratio, *r*?
Note that -0.5 is the number that is being raised to the exponent. So each time the number goes up on the exponent, in essence you are multiplying it by -0.5.

Putting in -5 for the first term and -0.5 for the common ratio we get:

$$S = \frac{a_1}{(1-r)}$$

$$S = \frac{-5}{(1-(-0.5))} = \frac{-5}{1.5} = -\frac{10}{3}$$