Factoring a Trinomial

Part I: Factoring $x^2 + bx + c$

In biology, *Punnett squares* are used to show possible ways that traits can be passed from parents to their offspring.
Each parent has two genes for each trait. The letters representing the parent's genes are placed on the outside of the Punnett square. The letters inside the boxes show the possible gene combinations for their offspring.
The Punnett square shows the gene combinations for fur color in rabbits.

	G	*g*
G	*GG*	*Gg*
g	*Gg*	*gg*

• *G* represents the dominant gene for gray fur.
• *g* represents the recessive gene for white fur.
Notice that the Punnett square is similar to the model for multiplying binomials.

Since the trinomial comes from multiplying two first-degree binomials, let's review what happens when we multiply binomials using the FOIL method. Remember that to do factoring we will have to think about this process in reverse (you could say we want to 'de-FOIL' the trinomial).
Suppose we are given: $(x + 2)\ (x + 3)$
Using the FOIL method, we get: $(x + 2)(x + 3) = x^2 + 3x + 2x + 6$
Then, collecting like terms gives: $(x + 2)(x + 3) = x^2 + 5x + 6$

Now look at this and think about where the terms in the trinomial came from. Obviously the x^2 came from x times x. The interesting part is what happens with the other parts, the '+ 2' and the '+ 3'. The last term in the trinomial, the 6 in this case, came from **multiplying** the 2 and the 3. Where did the $5x$ in the middle come from? We got the $5x$ by **adding** the $2x$ and the $3x$ when we collected like terms.

We can state this as a **rule**:

If the coefficient of x^2 is one, then to factor the trinomial of the form $x^2 + bx + c$ you need to find two numbers that:
1. Multiply to give the constant term (which we call c)
2. Add to give the coefficient of x (which we call b)
This rule works even if there are minus signs in the quadratic expression (assuming that you remember how to add and multiply positive and negative numbers).

To factor a trinomial of the form x^2+bx+c, follow the steps below:

Step 1: Set up a product of two ( ) where each will hold two terms.
It will look like this: ( )( ).

Step 2: Find the factors that go in the first positions.
To get the *x* squared (which is the F in FOIL), we would have to have an *x* in the first positions in each ( ).
So it would look like this: (*x*)(*x*).

Step 3: Find the factors that go in the last positions.
The factors that would go in the last position would have to be two expressions such that their **product equals *c*** (the constant) and at the same time their **sum equals *b*** (number in front of *x* term).

Remark 1: As you are finding these factors, you have to consider the sign of the expressions:
If *c* is positive, your factors are going to both have the same sign depending on *b*'s sign.
If *c* is negative, your factors are going to have opposite signs depending on *b*'s sign.

Example 1: Factor the trinomial y^2-5y+6.

Note that this trinomial does not have a GCF.
So we go right into factoring the trinomial of the form x^2+bx+c.
Step 1: Set up a product of two ( ) where each will hold two terms.
It will look like this: ( )( )
Step 2: Find the factors that go in the first positions.
Since we have y squared as our first term, we will need the following: (*y*)(*y*)
Step 3: Find the factors that go in the last positions.
We need two numbers whose product is 6 and sum is -5. That would have to be -2 and -3.
Putting that into our factors we get: $y^2-5y+6=(y-2)(y-3)$
-2 and -3 are two numbers whose product is 6 and sum is -5
Note that if we would multiply this out, we would get the original trinomial.

Remember **"What numbers multiply to the last term and add to the middle term?"**

Part II: Factoring $ax^2 + bx + c$

A quadratic is more difficult to factor when the coefficient of the squared term is not 1, because that coefficient is mixed in with the other products from FOIL of the two binomials.

If you need to factor a trinomial such as $2x^2 + x - 3$, you have to think about what combinations could give the $2x^2$ as well as the other two terms. In this example the $2x^2$ must come from $(x)(2x)$, and the constant term might come from either (-1)(3) or (1)(-3). The hard part is figuring out which combination will give the correct middle term. This gets messy because all those coefficients will be mixed in with the middle term when you FOIL the binomials. To see what is going on, let's see what happens when we FOIL the following binomials:

$$(x-1)(2x+3) = 2x^2 + 3x - 2x - 3 = 2x^2 + x - 3$$

What happened? There are several significant things to notice:

1. The leading term in the trinomial (the $2x^2$) is just the product of the leading terms in the binomials.
2. The constant term in the trinomial (the - 3) is the product of the constant terms in the binomials (so far this is the same as in the case where the coefficient of x^2 is 1)
3. The middle term in the trinomial (the x) is the sum of the outer and inner products, which involves *all* the constants and coefficients in the binomials, in a messy way that is not always obvious by inspection.

Because 1 and 2 are relatively simple and 3 is complicated, it makes sense to think of the possible candidates that would satisfy conditions 1 and 2, and then test them in every possible combination by multiplying the resulting binomials to see if you get the correct middle term. This seems tedious, and indeed it can be if the numbers you are working with have a lot of factors, but in practice you usually have to try a few combinations only before you see what will work. As a demonstration, let's see the example:

Given: $2x^2 + x - 3$

We make a list of the possible factors of $2x^2$: The only choice is $(2x)(x)$.

Then we make a list of the possible factors of the constant term -3: it is either (1)(-3) or (-1)(3). (Notice that since we need a negative number, one factor must be negative and the other positive, but it doesn't matter which one so we have to try it both ways).

The possible factors of the trinomial are the binomials that we can make out of these possible factors, taken in every possible order. From these possibilities, we see that the candidate binomials are:

$(2x + 1)(x - 3)$
$(x + 1)(2x - 3)$
$(2x + 3)(x - 1)$
$(x + 3)(2x - 1)$

If we start multiplying these out, we will find that the third one works, and then we are finished. All you really need to check is to see if the sum of the outer and inner multiplications will give you the correct middle term, since we already know that we will get the correct first and last terms.

Rule 2: To factorize a trinomial of the form ax^2+bx+c, follow the steps listed below:

1. List all the possible ways to get the coefficient of x^2 (which we call a) by multiplying two numbers
2. List all the possible ways to get the constant term (which we call c) by multiplying two numbers
3. Try all possible combinations of these to see which ones give the correct middle term
4. Set up a product of two ( ) where each will hold two terms.
   It will look like this ( )( )
5. Use trial and error to find the factors needed.

The factors of a will go in the first terms of the binomials and the factors of c will go in the last terms of the binomials.

Remark

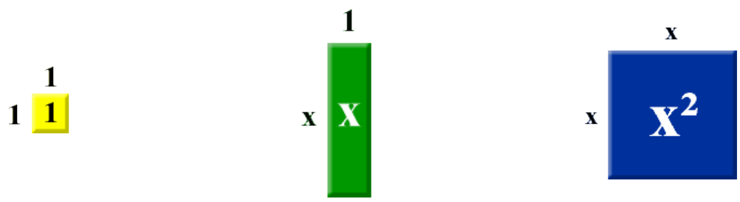
- Don't forget that the number itself times 1 is a possibility
- If the number (a or c) is negative, remember to try the plus and minus signs both ways

Note:

The trick of factorization is to get the right combination of factors. You can check this by applying the FOIL method. If your product comes out to be the trinomial you started with, you have the right combination of factors. If the product does not come out to be the given trinomial, then you need to try again.

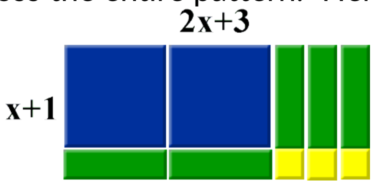
Rule 3: Factoring using Algebra Tiles

Key:



Factor $2x^2 + 5x + 3$

Using algebra tiles, build a rectangle containing the tiles specified in this problem (2 x^2-tiles, 5 x-tiles and 3 1-tiles). Remember that the lines between the tiles within your pattern must be completely vertical or horizontal across the entire pattern. Here is one possible arrangement:



After the pattern is established, it can be seen that the top edge of the pattern (the length) is composed of tiles with dimensions $2x + 3$. The side edge of the pattern (the width) is composed of tiles with dimensions $x + 1$.

Consequently, $\mathbf{2x^2 + 5x + 3 = (2x + 3)(x + 1)}$.