Exponential Functions

We already dealt with algebraic functions which include polynomial functions and rational functions.
The exponential function is a non-algebraic function, it is a transcendental function.

Definition1: The exponential function with base a is denoted by $f(x)=a^x$ where $a>0$, $a\neq 1$, and x is any real number.

Remarks:

1) a is positive $\Rightarrow$ the function will always be positive
2) The base $a=1$ is excluded because it yields $f(x)=1^x=1$. This is a constant function, not an exponential function.

How do you know if the Function is Exponential?

The key to determine if you have an exponential function is to check if the function has a constant growth factor. So you need to determine the ratios f(x+1): f(x) and see if they are all the same; If so, your function is exponential and this constant is the value of a. To find b, find the output for x = 0. Remember, b is your starting value. If that output is not given, you can determine b by solving $f(x)=b\bullet a^x$ for b. Doing so, we get $b=\dfrac{f(x)}{a^x}$

Example 1: Determine if f is an exponential function. If so, find its equation. If not, briefly justify why.

x	0	1	2	3	4
f(x)	8	1.6	0.32	0.064	0.0128

To decide if this is an exponential function, we need to see if the growth factor is constant.

$$\frac{f(1)}{f(0)}=\frac{1.6}{8}=0.2$$

$$\frac{f(2)}{f(1)}=\frac{0.32}{1.6}=0.2$$

$$\frac{f(3)}{f(2)}=\frac{0.064}{0.32}=0.2$$

$$\frac{f(4)}{f(3)}=\frac{0.0128}{0.064}=0.2$$

Since the ratio $\dfrac{f(x+1)}{f(x)}$ is constant, this is an exponential function with a = 0.2. To find the value of b, notice that f(0) = 8. So this is the exponential function $f(x)=8\bullet 0.2^x$

To evaluate a^x for any real number x we simply substitute. Taking into consideration all the properties of exponents.

Properties of Exponents

For working with exponential functions, the following properties of exponents are useful.
Let x and ybe positive numbers

1) $a^0 = 1$

2) $a^x \bullet a^y = a^{x+y}$

3) $\left(a^x\right)^y = a^{x \bullet y}$

4) $\dfrac{a^x}{a^y} = a^{x-y}$

5) $(ab)^x = a^x \bullet b^x$

6) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

7) $\dfrac{1}{a^x} = a^{-x}$

Example 2: Evaluate $y = 9^x$ if $x = 0.5$

$y = 9^{0.5} \Rightarrow y = 3$

Graphing an exponential function

The graphs of all exponential functions have similar characteristics. To graph any exponential function we have to:

- Specify the domain and the range
- Check for symmetry, shifting, reflecting, and translation
- Construct the table of values for $x \,\&\, y$ (at least 4 points)
- Find the asymptote
- Draw a Cartesian system
- Plot and join the points

For the function $f(x) = a^x$, the x-axis is an asymptote. (An asymptote is a line that approaches a given curve arbitrarily closely.)

Example 3: Consider the function $f(x) = 2^x$.

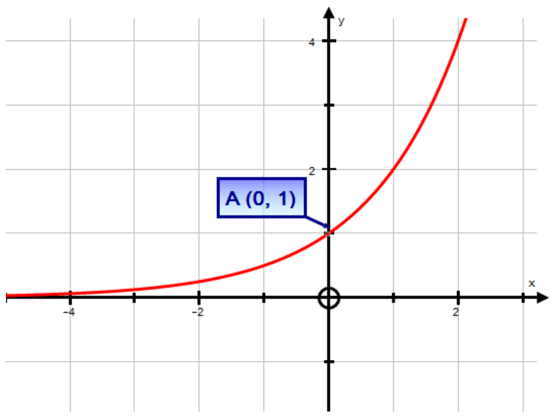
Domain: $x \in \mathbb{R}$

Range: $f(x) > 0 \Rightarrow y = 0$ is a horizontal asymptote

In this case, we have an exponential function with base 2. Some typical values for this function would be:

x	-2	-1	0	1	2	3
f(x)	1/4	1/2	1	2	4	8

Here is the graph of $y = 2^x$.



Example 4: Graph the functions $f(x) = 4^x$, $g(x) = 4^x + 3$, $h(x) = 4^{x-4}$ and $k(x) = 4^{x-1} - 2$

Let us make the discussion for the function $k(x) = 4^{x-1} - 2$ (same for the other functions)

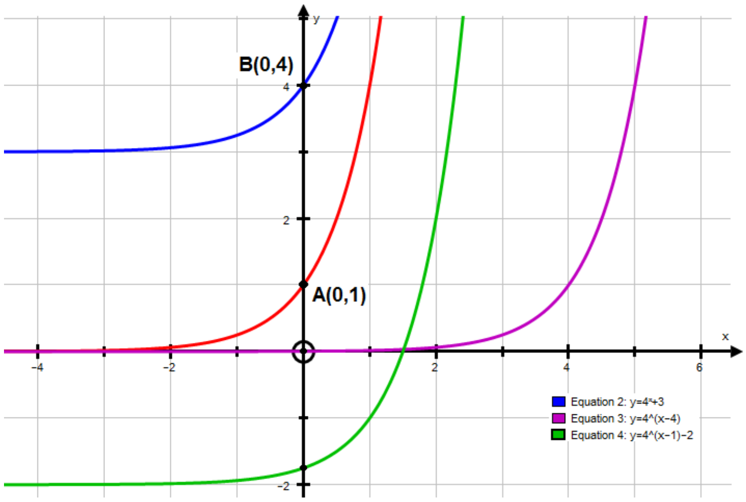
Domain: x is any real number
Range: $y > -2$

$\Rightarrow y = -2$ is a horizontal asymptote

<u>Vertical Shift:</u>

$$\left.\begin{array}{l} y.............y+2 \\ y.............y-b \end{array}\right\} \Rightarrow b = -2$$

$\Rightarrow$ There is a vertical shift 2 units downwards compared with the basic graph $f(x) = 4^x$



<u>Horizontal Shift:</u>

$$\left.\begin{array}{l} x.............x-1 \\ x.............x-a \end{array}\right\} \Rightarrow a = 1$$

$\Rightarrow$ There is a horizontal shift 1 unit to the right compared with the basic graph $f(x) = 4^x$

Since there is a vertical shift 2 units downward $\Rightarrow$ the asymptote will be shifted from the x-axis to the line y=-2. (ie 2 units downwards)

Graphing them all in the same Cartesian system will give the diagram

Example 5: Simplify each expression using the laws of exponents

a) $(9^{2x+4})(81)^{x-1}\left(\frac{1}{9}\right)^{1-3x}$

$$\begin{aligned}
&\left(9^{2x+4}\right)(81)^{x-1}\left(\frac{1}{9}\right)^{1-3x} \\
&=\left(3^2\right)^{2x+4}\left(3^3\right)^{x-1}\left(3^{-1}\right)^{1-3x} \\
&=(3)^{4x+8}(3)^{3x-3}(3)^{3x-1} \\
&=(3)^{7x+5}(3)^{3x-1} \\
&=3^{7x+5+3x-1} \\
&=3^{10x+4}
\end{aligned}$$

b) $\left(\frac{8}{125}\right)^{x-1} \bullet \left(\frac{625}{16}\right)^{2x-3}$

$$\begin{aligned}
&\left(\frac{8}{125}\right)^{x-1} \bullet \left(\frac{625}{16}\right)^{2x-3} \\
&=\left[\left(\frac{2}{5}\right)^{3}\right]^{x-1} \bullet \left[\left(\frac{5}{2}\right)^{4}\right]^{2x-3} \\
&\left[\left(\frac{5}{2}\right)^{-3}\right]^{x-1} \bullet \left[\left(\frac{5}{2}\right)^{4}\right]^{2x-3} \\
&=\left(\frac{5}{2}\right)^{-3x+3} \bullet \left(\frac{5}{2}\right)^{8x-12} \\
&=\left(\frac{5}{2}\right)^{(-3x+3)+(8x-12)} \\
&=\left(\frac{5}{2}\right)^{5x-9}
\end{aligned}$$