Domain and Range of Functions

The domain and range of a function are the essence or foundation of algebraic equations and calculus formulas. Everyday uses include graphs, charts and maps.

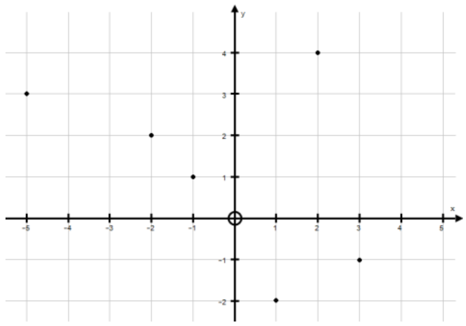
The domain of a function $y = f(x)$ is the set of all values of x for which the function is defined. In other words, a number $x = a$ is in the domain of a function f if $f(a)$ is a real number. For example, the domain of $f(x) = x^2$ is all real numbers since for any real number $x = a$, the value of $f(a) = a^2$ is also a real number.

Definition 1: A function is a set of ordered pairs (*x*,*y*) such that for each first element *x*, there always corresponds one and only one element *y*. The domain is the set of the first elements and the range is the term given to name the set of the second elements. The domain is referred to as the independent variable and the range as the dependent variable.

The domain is the first group or set of values being fed or input into a function and these values will serve as the x-axis of a graph or chart.

The range is the second group or set of values being fed or input into a function with these values serving as the y-axis of a graph or chart.
The domain and range can be clearly identified graphically

Example 1: State the domain and the range of
$\{(-1,1);(-2,2);(-5,3);(2,4);(3,-1);(1,-2)\}$
The points are plotted as shown.
The domain is the set of x values
$\Rightarrow Domain:\{-5,-1,-2,1,2,3\}$
The range is the set of y values $\Rightarrow Range:\{-1,-2,1,2,3,4\}$



It is very helpful to classify a function to determine its domain. For example, the function $f(x) = x^2$ belongs to a class of functions called **polynomial functions**. The domain of every polynomial function is all real numbers. Polynomial functions are discussed in great detail in Chapter 4.

Definition 2: The function $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ is a **polynomial function** of degree n where n is a nonnegative integer and $a_0, a_1, a_2, \ldots, a_n$ are real numbers.

The domain of every polynomial function is $(-\infty, \infty)$.

Many functions can have restricted domains. For example, the quotient of two polynomial functions is called a **rational function.** The rational function $f(x)=\frac{x+5}{x-4}$ is defined everywhere except when $x=4$ because the value $f(4)=\frac{9}{0}$ is undefined. Therefore, the domain of a rational function consists of all real numbers for which the denominator does not equal zero.

Definition 3: A **rational function** is a function of the form $f(x)=\frac{g(x)}{h(x)}$ where *g* and *h* are polynomial functions such that $h(x)\neq 0$.

The domain of a rational function is the set of all real numbers such that $h(x)\neq 0$.

Root functions can also have restricted domains. Consider the root function $f(x)=\sqrt{x+1}$. The number $x=3$ is in the domain because $f(3)=\sqrt{3+1}=\sqrt{4}=2$ is a real number. However, -5 is not in the domain because $f(-5)=\sqrt{-5+1}=\sqrt{-4}=2i$ is not a real number. Therefore, the domain of $f(x)=\sqrt{x+1}$ consists of all values of *x* for which the radicand is greater than or equal to zero. The domain of g is the solution to the inequality $x+1\geq 0$.

$$x+1\geq 0$$
$$x\geq -1 \quad \text{Subtract 1 from both sides.}$$

Therefore, the domain of f is $[-1,\infty)$. Root functions with roots that are odd numbers such as 3 or 5 **can have** negative radicands. Therefore, the domain of a root function of the form $f(x)=\sqrt[n]{g(x)}$ where *n* is an odd positive integer consists of all real numbers for which $g(x)$ is defined.

Definition 4: **Root Function:** The function $f(x)=\sqrt[n]{g(x)}$ is a **root function** where n is a positive integer.

- If n is *even*, the domain is the solution to the inequality $g(x)\geq 0$.
- If n is *odd*, the domain is the set of all real numbers for which $g(x)$ is defined.

Below is a quick guide for finding the domain of 3 specific types of functions:

Class of function	Form	Domain
Polynomial Functions	$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$	Domain is $(-\infty, \infty)$
Rational Functions	$f(x) = \dfrac{g(x)}{h(x)}$ where g and $h \neq 0$ are polynomial functions	Domain is all real numbers such that $h(x) \neq 0$
Root Functions	$f(x) = \sqrt[n]{g(x)}$, where $g(x)$ is a function and *n* is a positive integer	1. If *n* is even, the domain is the solution to the inequality $g(x) \geq 0$ 2. If *n* is odd, the domain is the set of all real numbers for which g is defined.

Finding Ranges: To find the range of a function, plug in the domain and find the minimum and maximum or solve for x and check the restrictions on the y – values. If the function can become indefinitely big, it will be to positive infinity. Similarly, if it can become indefinitely small, it will be to negative infinity.

We can easily find the range of a function if any of the two following cases is applicable:

1. If it has only absolute values or squares and square roots, it can never be negative
2. If it is a quotient and the numerator is constant, it can never be equal to 0