Demoivre's Theorem and nth Roots

The trigonometric form is used to raise a complex number to a power. To accomplish this, consider repeated use of the multiplication rule.

$$z = r(\cos\theta + i\sin\theta)$$

$$z^2 = r(\cos\theta + i\sin\theta)r(\cos\theta + i\sin\theta) = r^2(\cos 2\theta + i\sin 2\theta)$$

$$z^3 = r(\cos\theta + i\sin\theta)r^2(\cos 2\theta + i\sin 2\theta) = r^3(\cos 3\theta + i\sin 3\theta)$$

$$z^4 = r^4(\cos 4\theta + i\sin 4\theta)$$

$$z^5 = r^5(\cos 5\theta + i\sin 5\theta)$$

This pattern leads to a very important theorem, which is named after the French mathematician Abraham De Moivre (1667-1754)

Theorem 1: De Moivre's Theorem: If $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is a positive integer, then:

$$z^n = \left[r(\cos\theta + i\sin\theta)\right]^n$$
$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Example 1: Compute $(3+3i)^5$

Of course we could just do this by multiplying the number out, but this would be time consuming. Instead we can convert to exponential form and then use DeMoivre's Theorem
We need to find the trigonometric form of $3+3i$

$$r = \sqrt{9+9} = 3\sqrt{2}$$
$$\tan\theta = \frac{3}{3} \Rightarrow Argz = \frac{\pi}{4}$$

$$3+3i = 3\sqrt{2}cis\frac{\pi}{4}$$
$$\Rightarrow (3+3i)^5$$
$$= \left(3\sqrt{2}cis\frac{\pi}{4}\right)^5$$
$$= \left(3\sqrt{2}\right)^5 cis5\left(\frac{\pi}{4}\right)$$
$$= 972\sqrt{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$
$$= -972 - 972i$$

A consequence of the Fundamental Theorem of Algebra is that a polynomial equation of degree n has n solutions in the complex number system. So, the equation $x^4 = 1$ has four solutions, and in this particular case you can find the four solutions by factoring and using Quadratic Formula.

$$x^4 = 1$$
$$\Rightarrow x^4 - 1 = 0$$
$$\Rightarrow (x^2 - 1)(x^2 + 1) = 0$$
$$\Rightarrow (x-1)(x+1)(x-i)(x+i) = 0$$
$$\therefore x = \pm 1 \quad and \quad x = \pm i$$

The complex number $u = a + bi$ is an n^{th} root of the complex number z if $z = u^n = (a+bi)^n$

To find a formula for an n^{th} root of the complex number, let u be the n^{th} root of z, where:

$u = s(\cos\beta + i\sin\beta)$ and $z = r(\cos\theta + i\sin\theta)$

By De Moivre's theorem and the fact that $u^n = z$, we have:

$$s^n(\cos n\beta + i\sin n\beta) = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow s^n \, cis(n\beta) = r\, cis(\theta)$$

$$\Rightarrow \begin{cases} s^n = r \\ n\beta = \theta + 2\pi k \end{cases}$$

$$\therefore \begin{cases} s = \sqrt[n]{r} \\ \beta = \dfrac{\theta + 2\pi k}{n} \end{cases}$$

Theorem 2: For a positive integer n, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly n distinct n^{th} roots given by $w_0, w_1, \dots, w_{n-1}$:

These roots, for θ in radians are:

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \quad where \quad k = 0,1,2,3,4,\dots\dots,n-1$$

or equivalently, for θ in degrees:

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right] \quad where \quad k = 0,1,2,3,4,\dots\dots,n-1$$

The n^{th} roots of z all have absolute value $\sqrt[n]{r}$ and hence their geometric representations lie on a circle of radius $\sqrt[n]{r}$ with center at the origin. Moreover, they are equally spaced around the circle.

Remarks:

1) When k exceeds n-1, the roots begin to repeat. For instance, if k=n, the angle $\frac{\theta+2\pi n}{n}=\frac{\theta}{n}+2\pi$ is co terminal with $\frac{\theta}{n}$, which is obtained when k=0.

2) If we are looking for the n^{th} roots of an equation involving complex numbers, the roots will be $\frac{360^{\circ}}{n}$ apart i.e.

   - 2 roots will be 180° apart
   - 3 roots will be 120° apart
   - 4 roots will be 90° apart
   - 5 roots will be 72° apart etc