Definite Integrals

The value of the definite integral of a given function is a real number, depending on its lower and upper limits only, and is independent of the choice of the variable of integration, i.e

$\int_a^b f(x)\mathrm{d}x = \int_a^b f(y)\mathrm{d}y = \int_a^b f(t)\mathrm{d}t$

$\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$

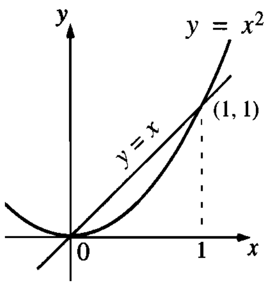
$\int_a^a f(x)\mathrm{d}x = \int_b^b f(x)\mathrm{d}x = \int_a^b 0\,\mathrm{d}x = 0$

Let $a \le c \le b$, then $\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$

Comparison of two integrals

If $f(x) \le g(x)\ \ \forall x \in (a,b)$, then $\int_a^b f(x)\mathrm{d}x \le \int_a^b g(x)\mathrm{d}x$

Example: $x^2 \le x$, for all $x \in (0,1)$; hence $\int_0^1 x^2\mathrm{d}x \le \int_0^1 x\,\mathrm{d}x$.



Property:

(i)If $f(-x) = f(x)$ **(Even Function)** then $\int_{-a}^a f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$

(ii)If $f(-x) = -f(x)$ **(Odd Function)** then $\int_{-a}^a f(x)\,\mathrm{d}x = 0$

Theorem 1: **Cauchy-Inequality for Integration**

If $f(x)$, $g(x)$ are continuous function on $[a,b]$, then

$$\left(\int_a^b f(x)g(x)\mathrm{d}x\right)^2 \le \left(\int_a^b [f(x)]^2\mathrm{d}x\right)\left(\int_a^b [g(x)]^2\mathrm{d}x\right)$$

Triangle Inequality for Integration: $\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x$

Continuity and Differentiability of a Definite Integral

Theorem 2: **Mean Value Theorem for Integral**: If $f(x)$ is continuous on [a,b] then there exists some *c* in [a,b] and $\int_a^b f(x)\mathrm{d}x = f(c)(b-a)$

Theorem 3: **Continuity of definite Integral**

If $f(t)$ is continuous on $[a,b]$ and let $A(x) = \int_a^x f(t)\mathrm{d}t\ \ \forall x \in [a,b]$ then $A(x)$ is continuous at each point *x* in $[a,b]$.

Theorem 4: **Fundamental Theorem of Calculus:** Suppose that $f(x)$ is continuous on the interval *a* ≤ *x* ≤ *b*, and let $F(x)$ be an antiderivative of $f(x)$. Then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example 1: Find:

1) $\int_2^4 \frac{3}{x} dx$

$$\int_2^4 \frac{3}{x} dx = 3[\ln x]_2^4 = 3(\ln 4 - \ln 2)$$

$$= 3 \times \ln \tfrac{4}{2} = 3 \times \ln 2 = \ln(2^3) = \ln 8$$

2) $\int_4^6 (2x+4)dx$

$$\int_4^6 (2x+4)dx = x^2 + 4x\Big|_4^6 = (6^2 + 4\cdot 6) - (4^2 + 4\cdot 4).$$

Example 2: Evaluate the definite integral $\int_{-4}^{-3} \frac{4}{1+2x} dx$

$$u = 1+2x \Rightarrow du = 2dx \Rightarrow \frac{du}{2} = dx$$

$$x = -4 \Rightarrow u = -7;\ x = -3 \Rightarrow u = -5$$

$$\int_{-4}^{-3} \frac{4}{1+2x} dx = \int_{-7}^{-5} \frac{4}{u} \times \frac{du}{2} = \int_{-7}^{-5} \frac{2}{u} du = 2\left[\ln|u|\right]_{-7}^{-5} = 2(\ln 5 - \ln 7) = 2\ln \tfrac{5}{7} = \ln\left(\tfrac{5}{7}\right)^2 = \ln \tfrac{25}{49}$$