Curve Sketching

First Derivative

Let *f* be continuous on an interval *I* and differentiable on the interior of *I*.
If $f'(x) > 0$ for all $x \in I$, then *f* is *increasing* on *I*.
If $f'(x) < 0$ for all $x \in I$, then *f* is *decreasing* on *I*.

Example 1: Determine the intervals where the function *f* is *increasing or decreasing*

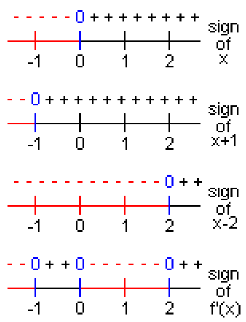
The function $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$ has first derivative

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$f'(x) = 12x(x^2 - x - 2)$$

$$f'(x) = 12x(x+1)(x-2)$$

Thus, $f(x)$ is increasing on $(-1,0) \cup (2,\infty)$ and decreasing on $(-\infty,-1) \cup (0,2)$.

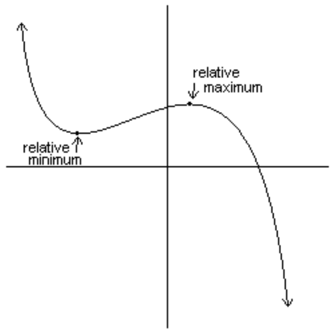


Relative Maxima and Minima
Relative extrema of *f* occur at critical points of *f*, values x_0 for which either $f'(x_0) = 0$ or $f'(x_0)$ is undefined.

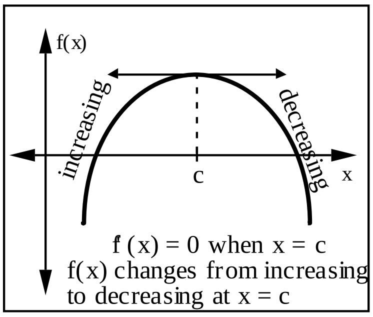
First Derivative Test

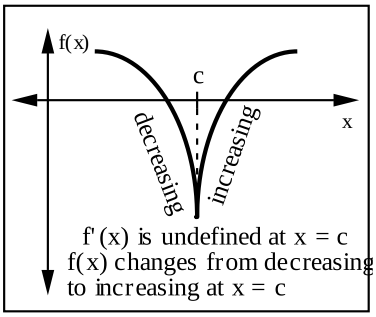
Suppose *f* is continuous at a critical point x_0.

1) If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then *f* has a relative maximum at x_0.
2) If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then *f* has a relative minimum at x_0.
3) If $f'(x)$ has the same sign on both an open interval extending left from x_0 and an open interval extending right from x_0, then *f* does not have a relative extremum at x_0.



In summary, relative extrema occur where $f'(x)$ changes sign.



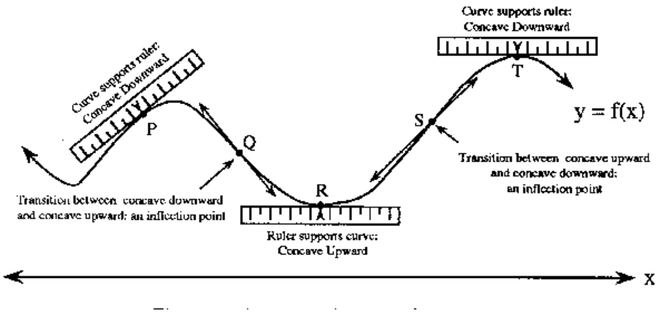


Concavity and the Second Derivative Test

The Second Derivative Test provides a means of classifying relative extreme values by using the sign of the second derivative at the critical number. To appreciate this test, it is first necessary to understand the concept of concavity.

The graph of a function *f* is **concave upward** at the point (*c*, *f*(*c*)) if *f*'(*c*) exists and if for all *x* in some open interval containing *c*, the point (*x*, *f*(*x*)) on the graph of *f* lies above the corresponding point on the graph of the tangent line to *f* at *c*. This is expressed by the inequality $f(x) > \left[f(c) + f'(c)(x-c) \right]$ for all *x* in some open interval containing *c*. Imagine holding a ruler along the tangent line through the point (*c*, *f*(*c*)): if the ruler *supports the graph of* f near (*c*, *f*(*c*)), then the graph of the function is concave upward.

The graph of a function *f* is **concave downward** at the point (*c*, *f*(*c*))$"> if *f*'(*c*) exists and if for all *x* in some open interval containing *c*, the point (*x*, *f*(*x*)) on the graph of *f* lies below the corresponding point on the graph of the tangent line to *f* at *c*. This is expressed by the inequality $f(x) < \left[f(c) + f'(c)(x-c) \right]$ for all *x* in some open interval containing *c*. In this situation *the graph of* f *supports the ruler*. This is pictured below:



Concavity and the Second Derivative
The important result that relates the concavity of the graph of a function to its derivatives is the following one:

Theorem 1: <u>Concavity Theorem:</u> If the function f is twice differentiable at x=c, then the graph of f is concave upward at $(c, f(c))$ if $f''(c) > 0$ and concave downward if $f''(c) < 0$

Inflection Points

Notice in the example above, that the concavity of the graph of *f* changes sign at x = 1. Points on the graph of *f* where the concavity changes from up-to-down or down-to-up are called **inflection points** of the graph. The following result connects the concept of inflection point to the derivatives properties of the function:

Theorem 2: **Inflection point Theorem:** If $f'(c)$ exists and $f''(c)$ changes sign at x=c, then the point $(c, f(c))$ is an inflection point of the graph of f. If $f''(c)$ exists at the inflection point, then $f''(c)=0$

If we return to our example, where $f(x)=x^3-3x^2+x-2$, the INFLECTION POINT THEOREM verifies that the graph of *f* has an inflection point at *x* = 1, since *f*''(1) = 0.

The Second derivative Test: *Suppose that c is a critical point at which* $f'(c)=0$, that $f'(x)$ exists in a neighborhood of c, and that $f''(c)$ exists. Then f has a relative maximum value at c if $f''(c)<0$ and a relative minimum value at c if $f''(c)>0$. If $f''(c)=0$, then the test is not informative.

To graph a polynomial function follow the steps listed below:

Step1: State the domain.

Step2: Find f'(x)

Step3: Construct a table of signs for f'(x) and interpret the results.
(Determine all relative and absolute maximum and minimum values and the intervals on which the function is increasing (↑), decreasing (↓))

Step4: Find f''(x)

Step5: Construct a table of signs for f''(x) and interpret the results.
(Determine all inflection points and the intervals on which the function is concave up (⋃), and concave down (⋂))

Step6: Determine *x*- and *y*-intercepts