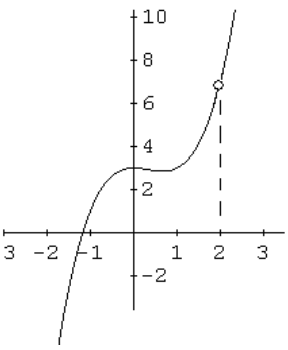
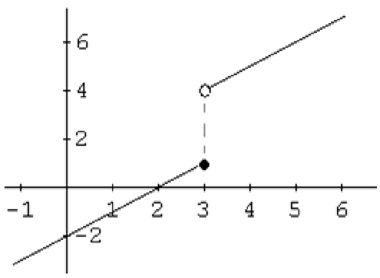
Continuity

In Mathematics the term continuous has much the same meaning as it has in everyday usage. Informally, to say that a function f is continuous at x=c means that there is no interruption in the graph of f at c. That is its graph is unbroken at c and there are no holes, jumps or gaps. The diagrams below identify three values of x at which the graph of f is not continuous. At all other points in the interval (a,b), the graph of f is uninterrupted and continuous.

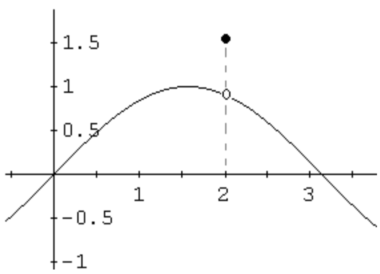
Discontinuous because it has a hole at x=2



Discontinuous because there is a jump at x=3

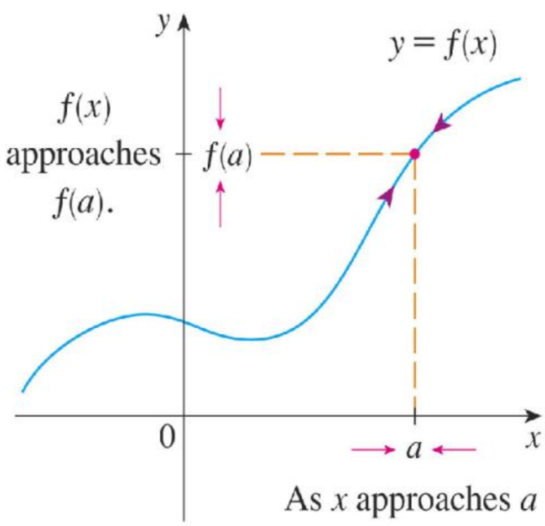


Discontinuous at x=2 because there is a gap



Definition 1: **Continuity at a point:** Let *f* be a function that is defined for all *x* in some open interval containing *a*. A function f(x) is continuous at x = a if the following three conditions are met:

1) $f(a)$ is defined (A function value exists at *x*=*a*.)
2) $\lim_{x \to a} f(x)$ exists (A limit value exists as you approach *x*=*a*.)
3) $\lim_{x \to a} f(x) = f(a)$ (The function value equals the limit value at *x*=*a*.)



Example 1: Show that the function $f(x)=\frac{\sqrt{x^2-x+1}}{x-5}$ is continuous at $x=-3$.

$$f(-3)=\frac{\sqrt{(-3)^2-(-3)=1}}{(-3)-5}=-\frac{\sqrt{13}}{8}$$

$\Rightarrow f(c)$ is defined.

$$\lim_{x\to -3} f(x)=\lim_{x\to -3}\frac{\sqrt{x^2-x+1}}{x-5}$$

$$=\frac{\lim_{x\to -3}\sqrt{x^2-x+1}}{\lim_{x\to -3}(x-5)} \qquad \text{limit of a quotient}$$

$$=\frac{\sqrt{\lim_{x\to -3}(x^2-x+1)}}{\lim_{x\to -3}(x-5)} \qquad \text{limit of a root}$$

$$=\frac{\sqrt{(-3)^2-(-3)+1}}{(-3)-5}$$

$$=-\frac{\sqrt{13}}{8}$$

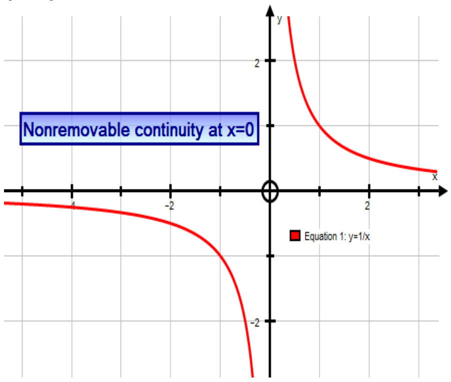
$\Rightarrow \lim_{x\to c} f(x)$ exists.

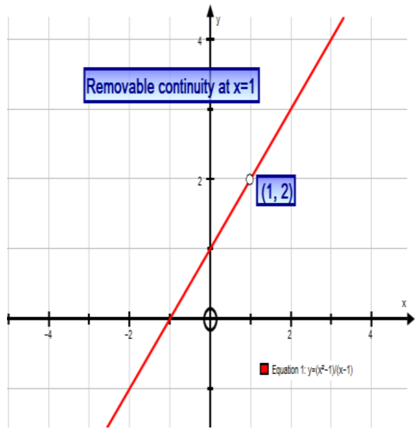
Therefore, $\lim_{x\to -3} f(x)=f(-3)$ and f is continuous at $x=-3$.

Consider an open interval I that contains a real number c. If a function f is defined on I (except possibly at c), and f is not continuous at c, then f is said to have a discontinuity at c. Discontinuity fall into two categories: **removable and nonremovable**. A discontinuity at c is removable if f can be made continuous by appropriately defining (or reducing) f(c).

- **Removable**: hole in graph at x = c
  Hole: factor and reduce, common factor is the hole

- **Nonremovable**: jump or break in graph
  Nonremovable (Vertical Asymptote): set denominator of reduced quotient equal to zero





<u>Continuity of Special Functions</u>

- Every **polynomial** function is continuous at every real number.
- Every **rational** function is continuous at every real number in its domain.
- Every **exponential** function is continuous at every real number.
- Every **logarithmic** function is continuous at every *positive* real number.
- $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every real number.
- $h(x) = \tan x$ is continuous at every real number in its domain.

<u>Continuity from the Left and Right</u>

- A function f is continuous from the right at $x = a$ provided that $\lim_{x \to a^+} f(x) = f(a)$.
- A function f is continuous from the right at $x = b$ provided that $\lim_{x \to b^-} f(x) = f(b)$.

<u>Continuity on an Interval</u>

1) A function f is said to be continuous on an **open** interval (a, b) provided that f is continuous at every value in the interval.

2) A function f is said to be continuous on a **closed** interval [a, b] provided that f is continuous from the **right at x = a(** $\lim_{x \to a^+} f(x) = f(a)$**)**, continuous from the **left at x = b(** $\lim_{x \to b^-} f(x) = f(b)$**)**, and continuous at every value in the open interval (a, b).

Properties of Continuous Functions
If the functions *f* and *g* are continuous at *x* = *c*, then each of the following functions are also continuous at *x* = *c*:

- ✓ Scalar Product: bf
- ✓ Sum and Difference: $f \pm g$
- ✓ Product: fg
- ✓ Quotient: $\frac{f}{g}$, *if* $g(c) \neq 0$

Properties of Composite Functions
If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

Theorem 1: **The Intermediate Value Theorem**
If the function *f* is continuous on the closed interval [*a*, *b*] and *k* is any number between *f(a)* and *f(b)*, then there exists at least one number *c* between *a* and *b* such that *f(c)* = *k*.