Combinations and Composition of Functions

Functions behave exactly as one would expect with regard to the four basic operations of algebra (addition, subtraction, multiplication, and division). When functions are combined by these operations, the domain of the new combined function is only the elements that were shared by the domains of the original functions.

Definition 1: Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the sum, difference, product, and quotient of f and g are defined as follows:

- The sum of two functions, *f* and *g*: $(f+g)_{(x)} = (f)_{(x)} + (g)_{(x)}$
- The difference of two functions *f* and *g*: $(f-g)_{(x)} = (f)_{(x)} - (g)_{(x)}$.
- The product of two functions *f* and *g*: $(f \bullet g)_{(x)} = (f)_{(x)} \bullet (g)_{(x)}$.
- The quotient of two functions *f* and *g*: $\left(\dfrac{f}{g}\right)_{(x)} = \dfrac{f_{(x)}}{g_{(x)}}$. If *g*(*x*) = 0, the quotient is undefined.

Example 1: Given $f(x) = 2x^2 - 3x + 4$ and $g(x) = x^3 + 4x$, find:

1) $(f+g)_{(x)}$

$$\begin{aligned}
&(f+g)_{(x)} \\
&= (f)_{(x)} + (g)_{(x)} \\
&= \left(2x^2 - 3x + 4\right) + \left(x^3 + 4x\right) \\
&= x^3 + 2x^2 + x + 4
\end{aligned}$$

2) $(f-g)_{(x)}$

$$\begin{aligned}
&(f-g)_{(x)} \\
&= (f)_{(x)} - (g)_{(x)} \\
&= \left(2x^2 - 3x + 4\right) - \left(x^3 + 4x\right) \\
&= -x^3 + 2x^2 - 7x + 4
\end{aligned}$$

3) $(f \bullet g)_{(x)}$

$$\begin{aligned}
&(f \bullet g)_{(x)} \\
&= (f)_{(x)} \bullet (g)_{(x)} \\
&= \left(2x^2 - 3x + 4\right) \bullet \left(x^3 + 4x\right) \\
&= \left(x^3 + 4x\right) \bullet \left(2x^2 - 3x + 4\right) \\
&= x^3 \bullet \left(2x^2 - 3x + 4\right) + 4x \bullet \left(2x^2 - 3x + 4\right) \\
&= 2x^5 - 3x^4 + 4x^3 + 8x^3 - 12x^2 + 16x \\
&= 2x^5 - 3x^4 + 12x^3 - 12x^2 + 16x
\end{aligned}$$

4) $\left(\dfrac{f}{g}\right)_{(x)}$

$$\begin{aligned}
&\left(\frac{f}{g}\right)_{(x)} \\
&= \frac{f_{(x)}}{g_{(x)}} \\
&= \frac{2x^2 - 3x + 4}{x^3 + 4x} \qquad x \neq 0
\end{aligned}$$

5) $(f+g)_{(-1)}$

$$(f+g)_{(-1)}$$
$$=(f)_{(-1)}+(g)_{(-1)}$$
$$=\left(2(-1)^2-3(-1)+4\right)+\left((-1)^3+4(-1)\right)$$
$$=9-5$$
$$=4$$

6) $(f-g)_{(2)}$

$$(f-g)_{(-2)}$$
$$=(f)_{(-2)}-(g)_{(-2)}$$
$$=-(-2)^3+2(-2)^2-7(-2)+4$$
$$=8+8+14+4$$
$$=34$$

There is one more way that functions can be combined. The 5th operation is called the composition of two functions.
Suppose we are given two functions f and g such that the range of g is contained in the domain of f so that the output of g can be used as input for f.

Definition 2: The composition of the functions $f_{(x)}$ and $g_{(x)}$ is symbolized by $(f\circ g)_{(x)}$. It is equivalent to $f\left(g_{(x)}\right)$. It is read " f of g of x."
The concept is simple. First, the value of g at x is taken, and then the value of f at the value of g(x) is taken.

The composition of $f_{(x)}$ with $g_{(x)}$: $(f\circ g)_{(x)}=f\left(g_{(x)}\right)$

The composition of $g_{(x)}$ with $f_{(x)}$: $(g\circ f)_{(x)}=g\left(f_{(x)}\right)$

Example 2: Let $f_{(x)}=x-1$ *and* $g_{(x)}=2x-3$.

1) $(f\circ g)_{(x)}=f\left(g_{(x)}\right)=f(2x-3)=(2x-3)-1=2x-4$

2) $(g\circ f)_{(x)}=g\left(f_{(x)}\right)=g(x-1)=2(x-1)-3=2x-2-3=2x-5$

Definition 3: Decomposition of functions: If a formula for $(f\circ g)_{(x)}$ is given then the process of finding the formulas for f and g is called decomposition.

Example 3: Write the function $h(x)=\dfrac{1}{(-x+6)^2}$ as a composition of two functions.

One way to write h as a composition of two functions is to take the inner function to be $g(x)=-x+6$ and the outer function to be: $f(x)=\dfrac{1}{x^2}$

Then you can write: $h(x)=\dfrac{1}{(-x+6)^2}=f(-x+6)=f(g(x))$