Basic Differentiation Rules

The Constant Rule

Let $f(x)=c$, by the limit definition of the derivative,

$$\frac{d}{dx}[c]=f'(x)$$
$$=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{c-c}{\Delta x}$$
$$=\lim_{\Delta x\to 0}0$$
$$=0$$

Theorem 1: The derivative of a constant function $f(x)=c$ is 0. That is, if c is a real number, then $\frac{d}{dx}[c]=0$

The Power Rule: If n is a positive integer greater than 1, then the binomial expansion produces

$$\frac{d}{dx}\left[x^n\right]=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^n-x^n}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{\cancel{x^n}+nx^{n-1}(\Delta x)+\frac{n(n-1)x^{n-2}}{2}(\Delta x)^2+\ldots+(\Delta x)^n-\cancel{x^n}}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\left[nx^{n-1}+\frac{n(n-1)x^{n-2}}{2}(\Delta x)^2+\ldots+(\Delta x)^{n-1}\right]$$
$$=nx^{n-1}+0+\ldots+0$$
$$=nx^{n-1}$$

Theorem 2: If n is a rational number, then the function $f(x)=x^n$ is differentiable and $\frac{d}{dx}\left[x^n\right]=nx^{n-1}$

In particular, if $n=1$ then $f(x)=x$, using the power rule its derivative is $f'(x)=x^{1-1}=x^0=1$

Example 1: Find the derivative

1) $\frac{d}{dx}x^3=3x^2$
2) $\frac{d}{dx}x^{-3}=-3x^{-4}$

Basic Differentiation Rules

Suppose *c* and *n* are constants, and f *and* g are differentiable functions.

(1) $f(x) = cg(x)$

$$f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b - x} = \lim_{b \to x} \frac{cg(b) - cg(x)}{b - x} = c \lim_{b \to x} \frac{g(b) - g(x)}{b - x} = cg'(x)$$

$$f(x) = cg(x) \Rightarrow f'(x) = cg'(x)$$

(2) $f(x) = g(x) \pm k(x)$

$$f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b - x} = \lim_{b \to x} \frac{[g(b) \pm k(b)] - [g(x) \pm k(x)]}{b - x} =$$

$$\lim_{b \to x} \frac{g(b) - g(x)}{b - x} \pm \lim_{b \to x} \frac{k(b) - k(x)}{b - x} = g'(x) \pm k'(x)$$

$$f(x) = g(x) \pm k(x) \Rightarrow f'(x) = g'(x) \pm k'(x)$$

(3) $f(x) = g(x)k(x)$

$$f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b - x} = \lim_{b \to x} \frac{g(b)k(b) - g(x)k(x)}{b - x} =$$

$$\lim_{b \to x} \frac{g(b)k(b) - g(b)k(x) + g(b)k(x) - g(x)k(x)}{b - x} =$$

$$\left[\lim_{b \to x} g(b)\right]\left[\lim_{b \to x} \frac{k(b) - k(x)}{b - x}\right] + \left[\lim_{b \to x} k(x)\right]\left[\lim_{b \to x} \frac{g(b) - g(x)}{b - x}\right] =$$

$g(x)k'(x) + k(x)g'(x)$ (**Product Rule**)

$$f(x) = g(x)k(x) \Rightarrow f'(x) = g(x)k'(x) + k(x)g'(x)$$

(4) $f(x) = \dfrac{g(x)}{k(x)}$

$$f(x) = \frac{g(x)}{k(x)} \Rightarrow f(x)k(x) = g(x) \Rightarrow g'(x) = f(x)k'(x) + k(x)f'(x)$$

$$\Rightarrow f'(x) = \frac{g'(x) - f(x)k'(x)}{k(x)} = \frac{g'(x) - \left[\frac{g(x)}{k(x)}\right]k'(x)}{k(x)} = \frac{k(x)g'(x) - g(x)k'(x)}{[k(x)]^2}.$$

This derivative rule is called the **Quotient Rule.**

$$f(x) = \frac{g(x)}{k(x)} \Rightarrow f'(x) = \frac{k(x)g'(x) - g(x)k'(x)}{[k(x)]^2}$$

The table below provides a summary for the basic rules:

Function	Derivative
$f(x) = c$	$f'(x) = 0$
$f(x) = x$	$f'(x) = 1$
$f(x) = x^n$	$f'(x) = nx^{n-1}$
$f(x) = cx^n$	$f'(x) = c(nx^{n-1})$
u and v are functions	
$f(x) = u \pm v$	$f'(x) = u' \pm v'$
$f(x) = uv$	$f'(x) = u'v + v'u$
$f(x) = \frac{u}{v} \quad v \neq 0$	$f'(x) = \frac{u'v - v'u}{v^2}$