Arithmetic Sequences and Series

An arithmetic sequence is a sequence such that each successive term is obtained from the previous term by adding or subtracting a fixed number called a difference.

The sequence 4, 7, 10, 13, 16 ... is an example of an arithmetic sequence - the pattern is that we are always adding a fixed number of three to the previous term to get to the next term.

Definition 1: An *arithmetic series* (or *arithmetic progression*) is a sequence of numbers such that each number differs from the previous number by a constant amount, called the *common difference*.
If a_1 is the first term, a_n is the *nth* term, *d* is the common difference, *n* is the number of terms then:

$$a_n = a_1 + (n-1)d$$

$$d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = a_5 - a_4 = ... = a_n - a_{n-1}$$

Be careful: don't think that every sequence that has a pattern in addition is arithmetic. It is arithmetic if you are always adding the SAME number each time.

Example 1: Find the first five terms and the 15th term of the arithmetic sequence

$$a_n = 3 + (n-1)\left(\frac{1}{2}\right)$$

n is our term number, we plug the term number into the function to find the value of the term.

Let's see what we get for our first five terms:

$$a_n = 3 + (n-1)\left(\frac{1}{2}\right)$$

$n = 1 \Rightarrow \quad a_1 = 3 + (1-1)\left(\frac{1}{2}\right) = 3 + 0 = 3$

$n = 2 \Rightarrow \quad a_2 = 3 + (2-1)\left(\frac{1}{2}\right) = 3 + \frac{1}{2} = 3\frac{1}{2}$

$n = 3 \Rightarrow \quad a_3 = 3 + (3-1)\left(\frac{1}{2}\right) = 3 + 1 = 4$

$n = 4 \Rightarrow \quad a_4 = 3 + (4-1)\left(\frac{1}{2}\right) = 3 + \frac{3}{2} = 4\frac{1}{2}$

$n = 5 \Rightarrow \quad a_5 = 3 + (5-1)\left(\frac{1}{2}\right) = 3 + \frac{4}{2} = 5$

Now lets check out the 15th term:

$$a_{15} = 3 + (15-1)\left(\frac{1}{2}\right) = 3 + (14)\left(\frac{1}{2}\right) = 3 + 7 = 10$$

What was the common difference for this arithmetic sequence?

$$d = a_2 - a_1 = 3 - 3\frac{1}{2} = \frac{1}{2}$$

Example 2: Write a formula for the n^{th} term of the arithmetic sequence -10, -5, 0, 5, ...

We will use the n^{th} term formula for an arithmetic sequence, $a_n = a_1 + (n-1)d$, to help us with this problem.

Basically we need to find two things, the first term of the sequence a_1 and the common difference, d.

What is a_1, the first term of the sequence?

The first term of this sequence is -10 $\Rightarrow a_1 = -10$

What is *d*, the common difference?

$d = a_2 - a_1 = -10 + 5 = -5$

Substituting the values of a_1 and d *in the formula, we obtain:*

$a_n = a_1 + (n-1)d$

$a_n = -10 + (n-1)(5)$

$a_n = -10 + 5n - 5 = -15 + 5n$

Example 3: Calculate a_{100} for the arithmetic sequence

17, 22, 27, 32 ...

By subtracting adjacent terms, you find that the common difference, d, equals 5. Adding (100 -1) times 5 to the first term gives

a_{100} = 17 + (100-1) (5)

= 17 + 495 = 512

Example 4: The number 68 is a term in the arithmetic sequence with a_1 = 5 and d = 3. Which term is it?

In this case you know that a_n = 68, and you must find the term number, n. using formula,

$a_n = a_1 + (n - 1)\, d$

68 = 5 + (n – 1) (3) substituting into the formula

63 = (n – 1) (3)

21 = n -1

∴ n = 22

Theorem 1: If $a_n = a_1 + (n-1)d$ is an arithmetic sequence then the sum of the sequence is

$$S_n = \sum_{i=1}^{n} a_n = S_n = \frac{n}{2}(a_1 + a_n)$$

In other words, to find the sum of the first ***n*** terms of a series you need to know the first term (t_1), the last term (t_n) and the number of terms (*n*).

Definition 2: The Sum of the First *n* Terms of Arithmetic Sequence

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

S_n is the sum of the first n terms in a sequence
n is the number of terms you are adding up

a_1 is the first term of the sequence

a_n is the nth term of the sequence

Example 5: Given the A.P. 8, 2, -4, -10... find $\sum_{k=1}^{18} a_k$.

The above symbol is telling you to **find the sum** of each term **from term 1**, (replace *k* with 1 in the statement to the right of sigma to get a_1) **to term 18** (again, replacing k with 18 to get a_{18}). In other words:

$\sum_{k=1}^{18} a_k$ means ***sum the first 18 terms***

$a_1 = 8$

It is an A.P. $\Rightarrow a_n = a_1 + (n-1)d$

$$a_n = a_1 + (n-1)d$$
$$a_{18} = a_1 + (18-1)d$$
$$a_{18} = (8) + 17(-6)$$
$$a_{18} = 8 - 102$$
$$a_{18} = -94$$

Now that we know the value of the first term, the last term, and *n*, we can use the formula to find the sum of a set number of terms of a sequence: $S_n = \frac{n}{2}(a_1 + a_n)$

Just fill in the information you already know and solve.

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{18} = \frac{18}{2}(8 + (-94))$$
$$S_{18} = \frac{18}{2}(-86)$$
$$S_{18} = \frac{-1548}{2}$$
$$S_{18} = -774$$

Therefore, $\sum_{k=1}^{18} a_k = -774$.

Example 6: Add the first 29 terms of the A.P. shown here: 3, 11, 19, ...

We know that we are looking for the sum of the first 29 terms, therefore *n* = 29. Putting this into the formula we get: $S_{29} = \frac{29}{2}\left(a_1 + a_{29}\right)$

We already know the value of *n*, and we know the value of a_1 (which is 3). We just need to figure out the value of a_{29}.

$$a_n = a_1 + (n-1)d$$
$$a_{29} = a_1 + (29-1)d$$
$$a_{29} = (3) + 28(8)$$
$$a_{29} = 3 + 224$$
$$a_{29} = 227$$

So now that we know the value of a_{29} is 227, we can fill in the rest of the formula used to calculate the sum and solve. We get:

$$S_n = \frac{n}{2}\left(a_1 + a_n\right)$$
$$S_{29} = \frac{29}{2}(3 + 227)$$
$$S_{29} = \frac{29}{2}(230)$$
$$S_{29} = \frac{6670}{2}$$
$$S_{29} = 3335$$

Therefore the sum of the first 29 terms is 3335.

Example 7: Find the sum of the arithmetic series $\sum_{n=5}^{14}(-2n+3)$

What is the first term?

$$-2n + 3 = -2(5) + 3 = -7$$

What is the last term?

$$-2n + 3 = -2(14) + 3 = -25$$

How many terms are we summing up?

If you start at 5 and go all the way to 14, there will be 10 terms.

Putting in -7 for the first term, -25 for the last term, and 10 for *n*, we get:

$$S_n = \frac{n}{2}\left(a_1 + a_n\right)$$
$$\sum_{n=5}^{14}(-2n+3) = \frac{10}{2}(-7-25) = -160$$

Example 8: Find S_{10} for the sequence $a_n = 6n - 1$

The formula says we need to know n, the first term, and the nth term.
n = 10 since we are asked to find the sum of the first 10 terms

$a_1 = 6(1) - 1 = 6 - 1 = 5$

$a_{10} = 6(10) - 1 = 60 - 1 = 59$

Substituting these values into the equation gives

$$S_{10} = \frac{10}{2}(5 + 59) = 5(64) = 320$$

Example 9: Find the sum of the arithmetic series 3 + 6 + 9 + ... + 99.

What is the first term? 3

What is the last term? 99

How many terms are we summing up?

Because of the way it is written with the 3 dots, this one is a little bit trickier. You have to do a little figuring. We can use the *n*th term of an arithmetic sequence and solve for *n*, the number of terms in the sequence:

$a_n = a_1 + (n-1)d$

$99 = 3 + (n-1)3$

$99 = 3 + 3n - 3$

$99 = 3n$

$n = 33$

Putting in 3 for the first term, 99 for the last term, and 33 for *n*, we get:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_n = \frac{33}{2}(3 + 99) = 1683$$

Example 10: Find the sum of the first 21 terms of the sequence 3, 7, 11, 15, . . .

The formula says we need to know n, the first term, and the nth term.
n = 21 since we are asked to find the sum of the first 21 terms.
d=4

$$\Rightarrow S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$\Rightarrow S_n = \frac{21}{2}\left[2(3) + (21-1)(4)\right]$$

$$\Rightarrow S_n = 903$$