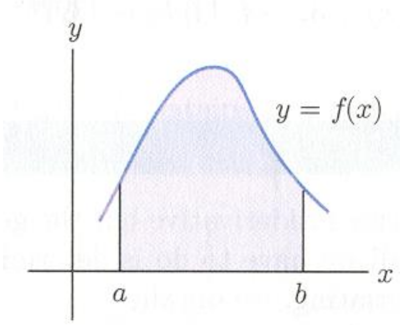
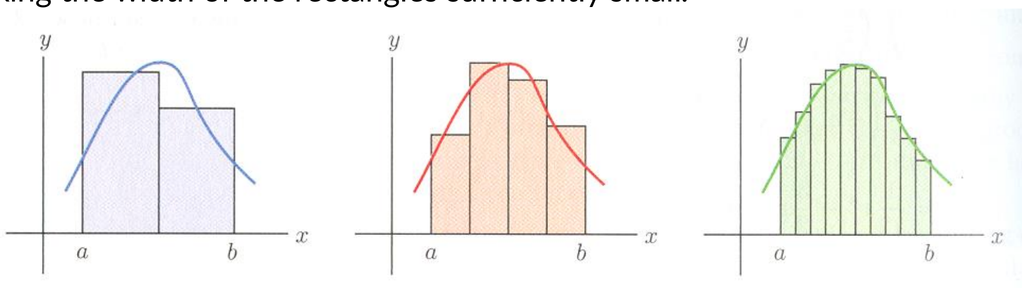
Area of a Region between Two Curves

If $f(x)$ is a continuous nonnegative function on the closed interval $a \le x \le b$, we refer to the area of the region shown below as the area under the graph of $f(x)$ from *a* to *b*.



Area under a graph
The computation of the area in the above figure is not trivial matter when the top boundary of the region is curved. However we can estimate the area to any desired degree of accuracy. The basic idea is to construct rectangles whose total area is approximately the same as the area to be computed. The area of each rectangle, of course is easy to compute.

The figure below shows three rectangular approximations of the area under a graph. When the rectangles are thin, the mismatch between the rectangles and the region under the graph is quite small. In general the rectangular approximation can be made as close as desired to the exact area simply by making the width of the rectangles sufficiently small.

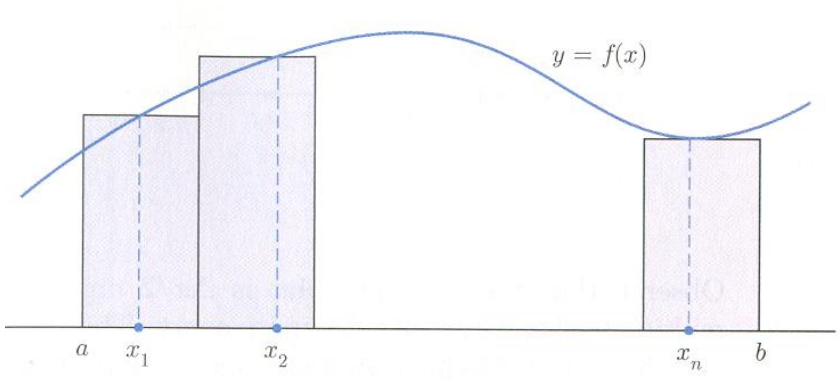


Approximating a region with rectangles

Given a continuous nonnegative function $f(x)$ on the interval $a \le x \le b$. Let us divide the *x*-axis interval into *n* equal subintervals, where *n* represents some positive integer. Such a subdivision is called a partition of the interval from *a* to *b*. Since the entire interval is of width $b-a$, so the width of each of the *n* subintervals is $(b-a)/n$. Therefore, we could denote this by Δx. That is

$$\Delta x = \frac{b-a}{n} \quad \text{(Width of one subinterval)}$$

In each subinterval, select a point. (Any point in the subinterval will do). Let x_1 be the point selected from the first subinterval, x_2 the point from the second subinterval, and so on. These points are use to form rectangles that approximate the region under the graph of $f(x)$. Construct the first rectangle with height $f(x_1)$ and the first subinterval as base, as in the figure below. The top of the rectangle touches the graph directly above x_1.



Rectangles with heights $f(x_1), f(x_2), \ldots, f(x_n)$

Notice that

[Area of the first rectangle] = [height][width] = $f(x_1)\Delta x$

The second rectangle rests on the second subinterval and has height $f(x_2)$. Thus

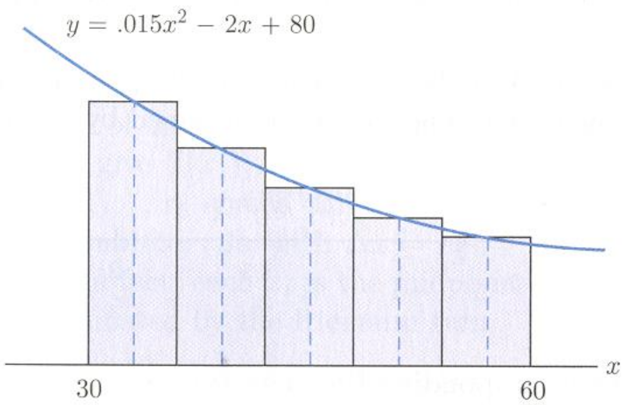
[Area of the second rectangle] = [height][width] = $f(x_2)\Delta x$

Continuing in this way, we construct *n* rectangles with a combined area of

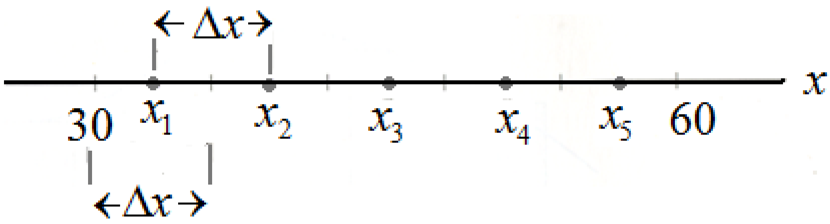
$$f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$
$$= \left[f(x_1) + f(x_2) + \ldots + f(x_n)\right]\Delta x$$

The above sum is called a Riemann sum. It provides an approximation to the area under the graph of $f(x)$ when $f(x)$ is nonnegative and continuous. In fact, as the number of subintervals increases indefinitely, the Riemann sums approach a limiting value, the ***area under the graph***.

Estimate the area under the graph of the marginal cost function $f(x) = 0.15x^2 - 2x + 80$ from $x = 30$ to $x = 60$. Use partitions of 5, 20, and 100 subintervals. Use the mid points of the subintervals as $x_1, x_2, \ldots, x_n$ to construct the rectangles.



The partition of $30 \le x \le 60$ with $n = 5$ is shown in the figure. The length of each subinterval is $\Delta x = \frac{60-30}{5} = 6$



A partition of the interval 30 ≤ x ≤ 60

Observe that the first midpoint is $\frac{\Delta x}{2}$ units from the left endpoints, and the midpoints themselves are Δx units apart.

The first midpoint is $x_1 = 30 + \frac{\Delta x}{2} = 30 + 3 = 33$

Subsequent midpoints are found by successively adding $\Delta x = 6$:

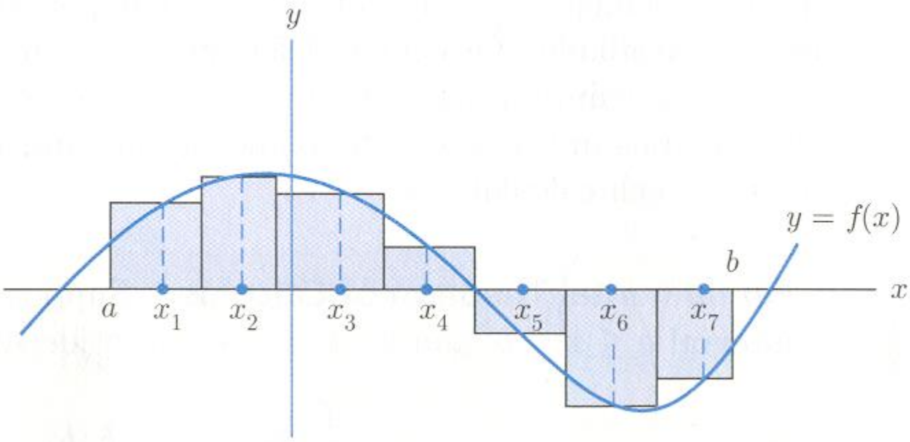
Midpoints: 33, 39, 45, 51, 57.

The corresponding estimate for the area under the graph of $f(x)$ is

$$f(33)\,\Delta x + f(39)\,\Delta x + f(45)\,\Delta x + f(51)\,\Delta x + f(57)\,\Delta x$$
$$= \left[f(33) + f(39) + f(45) + f(51) + f(57)\right]\Delta x$$
$$= [30.335 + 24.815 + 20.375 + 17.015 + 14.735] \cdot 6$$
$$= (107.275)(6)$$
$$= 643.65$$

A similar calculation with 20 subintervals produces an area estimate of 644.916. With 100 subintervals the estimate is 644.997.

In some case which $f(x)$ is negative at some points in the interval, we may also give a geometric interpretation of the definite integral. Consider the function $f(x)$ shown in the figure below. The figure shows a rectangular approximation of the region between the graph and the *x*-axis from *a* to *b*.



Consider a typical rectangle located above or below the selected point x_i. If $f(x_i)$ is nonnegative, the area of the rectangle equals $f(x_i)\Delta x$. In case $f(x_i)$ is negative, the area of the rectangle equals $(-f(x_i))\Delta x$. So the expression $f(x_i)\Delta x$ equals either the area of the corresponding rectangle or the negative of the area, according to whether $f(x_i)$ is nonnegative or negative, respectively.

In particular, the Riemann Sum

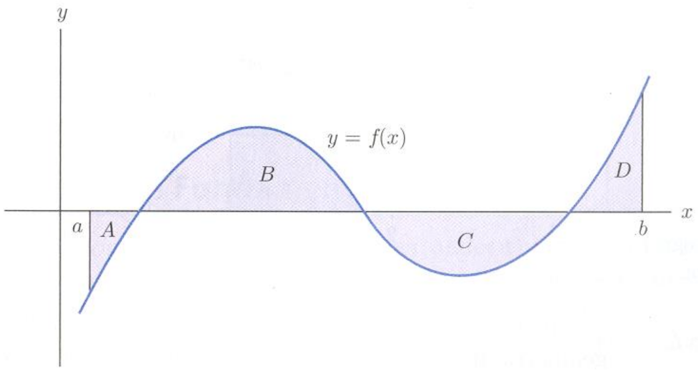
$$f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x = [f(x_1) + f(x_2) + \ldots + f(x_n)]\Delta x$$

is equal to the area of the rectangles above the *x*-axis minus the area of the rectangles below the *x*-axis. This gives us the following geometric interpretation of the definite integral.

Definition: Suppose that $f(x)$ is continuous on the interval $a \le x \le b$. then $\int_a^b f(x)dx$ is equal to the area above the *x*-axis bounded by the graph of $y = f(x)$ from $x = a$ to $x = b$ minus the area below the x-axis.

Referring to the figure below, we have

$$\int_a^b f(x)dx = [\text{area of B and D}] - [\text{area of A and C}]$$



Regions above and below the x-axis

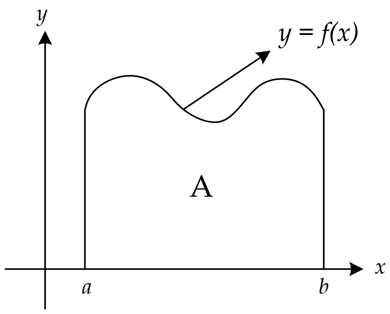
Recall the following theorem,

Fundamental theorem of calculus: Suppose that $f(x)$ is continuous on the interval $a \le x \le b$, and let $F(x)$ be an antiderivative of $f(x)$. Then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The following summarize the formula for area under the curve.

Area under the curve



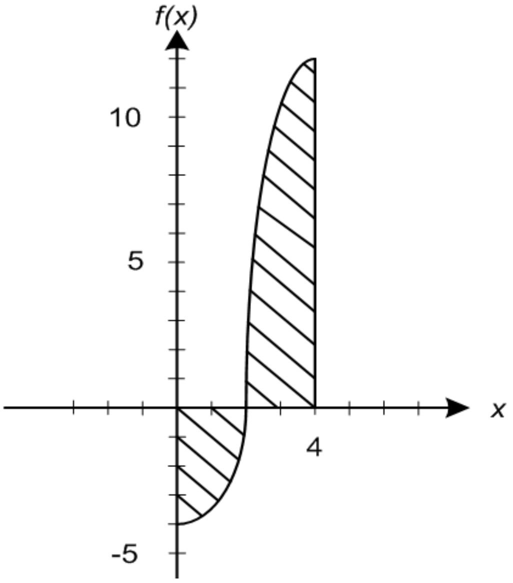
$$\therefore \quad A = \int_a^b f(x)\,dx$$
$$= f(b) - f(a)$$

Example 1: Find the area between x-axis and the graph of f, $f(x)=x^2-4$ from $x=0$ to $x=4$.

When, $x=0,\ f(x)=-4$

$x=4,\ f(x)=12$

$y=0,\ x=4$



$$A=\int_0^2 (x^2-4)\,dx+\int_2^4 (x^2-4)\,dx$$

$$=\left|\left[\frac{x^3}{3}-4x\right]_0^2\right|+\left|\left[\frac{x^3}{3}-4x\right]_2^4\right|$$

$$=\left|\left(\frac{8}{3}-8-0\right)\right|+\left|\left(\frac{64}{3}-16-\frac{8}{3}+8\right)\right|$$

$$=\left|-\frac{16}{3}\right|+\left|\frac{32}{3}\right|$$

$$=16$$

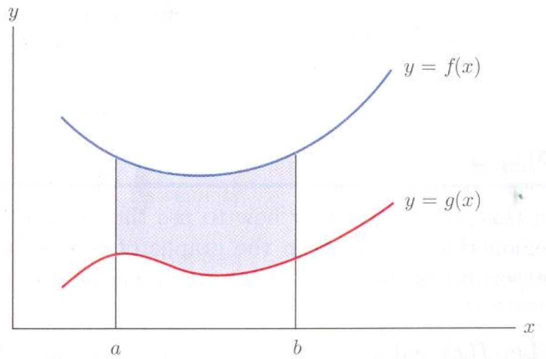
Area between Curves

Let's consider regions that are bounded both above and below by graphs of functions. Referring to Figure 5.8, we would like to find a simple expression for the area of the shaded region under the graph of $y=f(x)$ and above the graph of the $y=g(x)$ from $x=a$ to $x=b$. It is the region under the graph of $y=f(x)$ with the region under the graph of $y=g(x)$ taken away. Therefore,

$$[\text{Area of shaded region}] = [\text{area under } f(x)] - [\text{area under } g(x)]$$

$$=\int_a^b f(x)dx-\int_a^b g(x)dx$$

$$=\int_a^b [f(x)-g(x)]dx$$



Thus,

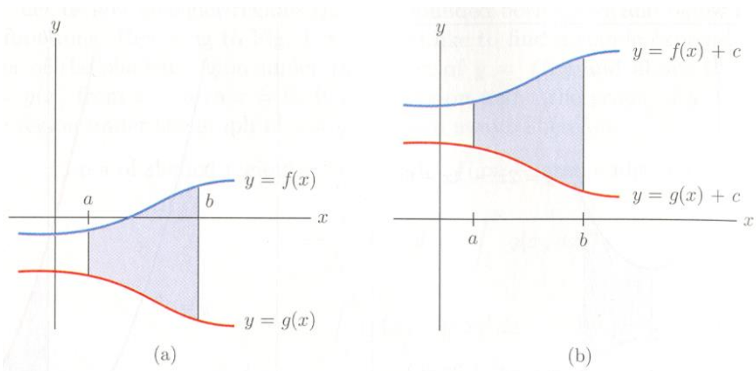
Area between two curves If $y=f(x)$ lies above $y=g(x)$ from $x=a$ to $x=b$, the area of the region between $f(x)$ and $g(x)$ from $x=a$ to $x=b$ is given by,

$$\int_a^b [f(x)-g(x)]dx$$

Referring to the above figure, both of the functions are nonnegative, considering the case where $f(x)$ and $g(x)$ are not always positive. Let us determine the area of the shaded region. Select some constant *c* such that the graphs of the functions $f(x)+c$ and $g(x)+c$ lie completely above the *x*-axis.

The region between them will have the same area as the original region. Using the rule as applied to nonnegative functions, we have:

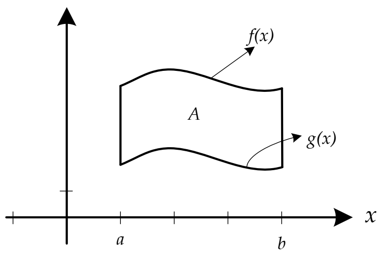
$$\text{[Area of the region]} = \int_a^b \left[\left(f(x)+c\right)-\left(g(x)+c\right)\right]dx$$
$$= \int_a^b \left[f(x)-g(x)\right]dx$$



Therefore, we see that our rule is valid for any functions $f(x)$ and $g(x)$ as long as the graph of $f(x)$ lies above the graph of $g(x)$ for all x from $x=a$ to $x=b$.

The following summarize the formula for area between curves.

Area between curves



$$\therefore \quad A = \int_a^b \left[f(x)-g(x)\right]dx$$