An Introduction to Sequences and Series

Sequences are to calculus like a calculator is to a scientist.

Consider a scientist doing an experiment; he is collecting data, let us say, every day. So, let x_1 to be the data collected the first day, x_2 be the data collected the second day, and so on.... and x_n is the data collected after n days. Clearly, we are generating a set of numbers with a very special characteristic: there is an order on the number, that is, we naturally have the first number, the second number, and so on.... A sequence is by definition a set of real numbers with natural order.

In general, a sequence is an ordered arrangement of numbers, figures, or objects.

Definition 1: Sequences of math are a string of numbers that are tied together with some sort of consistent rule, or set of rules, that determines the next number in the sequence.

Definition 2: The terms of a sequence are the output values or dependent variables.

In general, a_n represents the n^{th} term of a sequence.
a represents the functional or output value and n represents the input value of term number. For example, a_5 represents the 5th term of the sequence.

Consider the sequence $\{2^n\}_{n\geq 1}$. It is clear that we have $2<4<8<16<32<......$
The numbers are getting bigger and bigger
Now consider the sequence $\left\{\frac{1}{n}\right\}_{n\geq 1}$ .In this case, we have $1\geq\frac{1}{2}\geq\frac{1}{3}\geq\frac{1}{4}\geq\frac{1}{5}.....$
Notice that the numbers are getting smaller and smaller.

Definition 3: An *increasing sequence* is one where $a_n < a_{n+1}$ for all values of n.

Definition 4: A *decreasing sequence* is one where $a_n > a_{n+1}$ for all values of n.

Example 1 : Prove that the sequence $\left\{\frac{1}{n}\right\}_{n\geq 1}$ is decreasing.

Let $n\geq 1$. We have $n < n+1$. Therefore, $\frac{1}{n+1}<\frac{1}{n}$

$\Rightarrow\left\{\frac{1}{n}\right\}_{n\geq 1}$ is decreasing

Example 2: Prove that the sequence $\{2^n\}_{n\geq 1}$ is increasing.
Let $n\geq 1$. We have $2^{n+1}=2\bullet 2^n$. Since 2 > 1, then $1\bullet 2^n<2\bullet 2^n$, which gives $2^n<2^{n+1}$.

$\Rightarrow\{2^n\}_{n\geq 1}$ is increasing

Definition 5: An **arithmetic** sequence is a sequence such that each successive term is obtained from the previous term by adding or subtracting of a fixed number called a difference.

The sequence 4, 7, 10, 13, 16 ... is an example of an arithmetic sequence - the pattern is that we are always adding a fixed number "three" to the previous term to get to the next term.

Definition 6: A **geometric** sequence is a sequence such that each successive term is obtained from the previous term by multiplying by a fixed number called a ratio.

The sequence 5, 10, 20, 40, 80 ... is an example of a geometric sequence. The pattern is that we are always multiplying by a fixed number "2" to the previous term to get to the next term.

Definition 7: A basic **Fibonacci** sequence is when two numbers are added together to get the next number in the sequence.
1, 1, 2, 3, 5, 8, 13,.... is an example of a Fibonacci sequence where the starting numbers (or seeds) are 1 and 1, and we add the two previous numbers to get the next number in the sequence.

Definition 8: A **finite** sequence is a sequence whose domain consists of the set {1, 2, 3 ... *n*} or in other words the first *n* positive integers.

Definition 9: An infinite sequence is a sequence whose domain consists of the set {1, 2, 3 ...} or in other words all positive integers.

The Greek letter sigma is used to indicate a summation.
Summation notation is a shorthand way of saying take the sum of certain terms of a sequence.

Let's look at the following: $\sum_{i=1}^{n} a_i$

- i represents the term number or index of summation. Note that any variable can be used here.
- a_i represents the general term.
- 1 represents the lower limit of summation
- *n* represents the upper limit of summation.

Note that these numbers can be any integer. Basically, you will find the sum of the terms that start at the lower limit and go through the upper limit.

This expression would represent the sum of terms 1 through *n*:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + a_5 + ... + a_n$$

Definition 10: A series is a sum of terms of a sequence.

Example 3: Find: $\sum_{i=4}^{7} 3i$

$$\sum_{i=4}^{7} 3i = 3(4)+3(5)+3(6)+3(7) = 12+15+18+21 = 66$$

Definition 11: **Finite Series:** A finite series is a series that has a sum of a finite number of terms.

$\sum_{i=1}^{9} 2i+1$ is a finite series

Definition 12: **Infinite Series:** An infinite series is a series that has a sum of an infinite number of terms.
In an infinite series, the upper limit is infinity ... which means there is no upper bound.

$\sum_{i=1}^{\infty}(-4i)$ is an infinite series. $5+5+5+5+5+5+\ldots$ is another example
The three dots indicate that this pattern will keep continuing on and on.

Example 4: Find the sum of the series $\sum_{i=1}^{5}\left[(-1)^i\left(i^2\right)\right]$

You find the terms of the series in the same way that you find terms of a sequence. Plug the term number in for the given variable. So in this problem, wherever there is an i in the term, the term number will replace it.
The difference between this problem and a sequence problem is that you will be adding all of the terms together to get your end result.

We have to replace i by the values of 1 through 5.

Let's see what we get when we add up terms plugging in 1, 2, 3, 4, and 5 for i:

$$\sum_{i=1}^{5}\left[(-1)^i\left(i^2\right)\right]$$
$$=\left[(-1)^1\left(1^2\right)\right]+\left[(-1)^2\left(2^2\right)\right]+\left[(-1)^3\left(3^2\right)\right]+\left[(-1)^4\left(4^2\right)\right]+\left[(-1)^5\left(5^2\right)\right]$$
$$=-1+4-9+16-25$$
$$=-15$$