

Vectors in the Plane

A **scalar quantity** is a quantity that has the property of size or magnitude but does not possess the property of direction. Examples include: mass, length, time, density, energy, area, volume, and temperature. The real number associated with a scalar quantity is simply referred to as a **scalar**.

A **vector quantity** is a quantity that has both properties of magnitude and direction. Examples include: velocity, wind movement, momentum, force, and displacement.

Vector quantities can be represented geometrically by a **directed line segment** (or arrow) (written as \vec{v}) where the “tail” of the arrow is the **initial point** and the “tip” of the arrow is the **terminal point**. These directed line segments are referred to as **vectors**.



The **magnitude** of a vector $\|v\|$ (also called the **norm** of a vector) is the length of the directed line segment.

Definition 1: The component form of a vector with initial point $A(x_A, y_A)$ and terminal point $B(x_B, y_B)$ is:

$$\vec{AB} = \langle x_B - x_A, y_B - y_A \rangle = \langle v_1, v_2 \rangle = v$$

The magnitude (or length) of v is

$$\|v\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} = \sqrt{(v_1)^2 + (v_2)^2}$$

Example 1: Find the component form and the magnitude of the vector v that has initial point $A(4, -7)$ and terminal point $B(-1, 5)$

The components of $v = \langle v_1, v_2 \rangle$ are:

$$v_1 = x_B - x_A = -1 - 4 = -5$$

$$v_2 = y_B - y_A = 5 - (-7) = 12$$

So, $v = \langle -5, 12 \rangle$ and the magnitude of v is

$$\|v\| = \sqrt{(v_1)^2 + (v_2)^2} = \sqrt{(-5)^2 + (12)^2} = \sqrt{169} = 13$$

Notation of vectors: There are many ways to denote a vector. Four methods include:

- 1) For a vector with initial point P and terminal point Q we can write: \overrightarrow{PQ} .
- 2) Bold lowercase letters, such as \mathbf{a} , \mathbf{b} , \mathbf{u} , \mathbf{v} , or \mathbf{w} , are used when the endpoints are not specified. (In handwritten work we can write \vec{a} , \vec{b} , \vec{u} , \vec{v} , or \vec{w} respectively).
- 3) For a vector \mathbf{a} with initial point at the origin and terminal point at (a_1, a_2) we can denote the vector as $\mathbf{a} = \langle a_1, a_2 \rangle$. The numbers a_1 and a_2 are called the horizontal and vertical **components** of the vector \mathbf{a} , respectively.

Scalar Multiplication of vectors

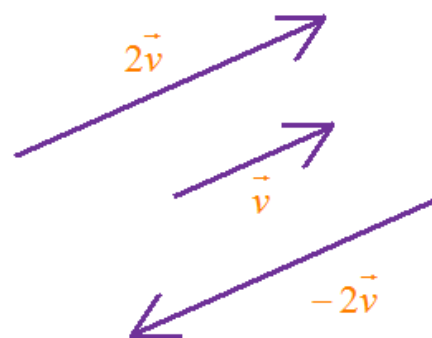
A scalar multiple of \mathbf{v} is given by $c\mathbf{v}$, where c is some real scalar.

This new vector, $c\mathbf{v}$, is $|c|$ times as long as \mathbf{v} .

If $c < 0$, then $c\mathbf{v}$ points in the opposite direction from the direction \mathbf{v} points in.

Scalar multiplication of vectors can be performed by multiplying the two components of the vector by the given scalar, for vector $\mathbf{v} = \langle v_1, v_2 \rangle$, and a scalar m ,

$$m\mathbf{v} = \langle m \cdot v_1, m \cdot v_2 \rangle$$



Example 2: Let $\mathbf{v} = \langle 6, -3 \rangle$, find:

- 1) $2\vec{v} = 2\langle 6, -3 \rangle = \langle 12, -6 \rangle$
- 2) $-2\vec{v} = -2\langle 6, -3 \rangle = \langle -12, 6 \rangle$

Remarks:

- 1) Two vectors have the *same direction* if they are parallel and point in the same direction.
- 2) Two vectors have *opposite directions* if they are parallel and point in opposite directions.
- 3) Two vectors are said to be **equivalent vectors** if they have the same magnitude and same direction. (a vector may be translated from one location to another, provided neither the magnitude nor the direction is changed.)

Example 3: Let \mathbf{v} be the vector from $(0, 0)$ to $(4, 5)$ and \mathbf{u} be the vector from $(2, 3)$ to $(6, 8)$. Show that $\mathbf{v} = \mathbf{u}$.

$$\|\mathbf{v}\| = \sqrt{(4-0)^2 + (5-0)^2} = \sqrt{41}$$

$$\|\mathbf{u}\| = \sqrt{(6-2)^2 + (8-3)^2} = \sqrt{41}$$

$$\Rightarrow \|\mathbf{v}\| = \|\mathbf{u}\|$$

Thus both vectors have the same magnitude.

Now we need to check if they have the same direction so we will find the slopes:

$$\left. \begin{aligned} m_u &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 0}{4 - 0} = \frac{5}{4} \\ m_v &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 3}{6 - 2} = \frac{5}{4} \end{aligned} \right\} \Rightarrow m_u = m_v$$

Both vectors also have the same slope, therefore they are equal.

Vector Addition

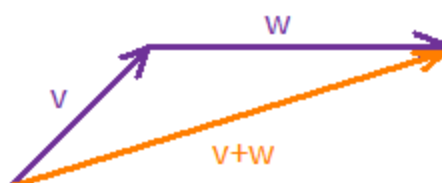
Vector addition can correspond to combined (or net) effects.

For example, if v and w are force vectors, the resultant vector $v + w$ represents net force.

Vector subtraction may be defined as follows: $v - w = v + (-w)$.

There are two ways we can graphically represent vector addition:

1) Triangle Law



To draw $v + w$, we place the tail of w at the head of v , and we draw an arrow from the tail of v to the head of w .

This may be better for representing sequential effects and displacements.

2) Parallelogram Law



To draw $v + w$, we draw v and w so that they have the same initial point, we construct the parallelogram (if any) that they determine, and we draw an arrow from the common initial point to the opposing corner of the parallelogram.

This may be better for representing simultaneous effects and net force.

Definition 2: Let $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$ be vectors, then the sum and difference of u and v are:

- 1) $u + v = \langle u_1 + v_1, u_2 + v_2 \rangle$
- 2) $u - v = \langle u_1 - v_1, u_2 - v_2 \rangle$

Example 4: Let $u = \langle -5, 2 \rangle$ and $v = \langle 6, -3 \rangle$, find:

- 1) $u + v = \langle -5 + 6, 2 + (-3) \rangle = \langle 1, -1 \rangle$
- 2) $u - v = \langle -5 - 6, 2 - (-3) \rangle = \langle -11, 5 \rangle$
- 3) $2u - v = 2\langle -5, 2 \rangle - \langle 6, -3 \rangle = \langle -10, 4 \rangle - \langle 6, -3 \rangle = \langle -16, 7 \rangle$

Properties of Addition and Scalar Multiples of Vectors: For vectors $u = \langle u_1, u_2 \rangle$, $v = \langle v_1, v_2 \rangle$, and scalars m and n :

- 1) $u + v = v + u$
- 2) $(u + v) + w = u + (v + w)$
- 3) $u + 0 = u$
- 4) $u + (-u) = 0$
- 5) $m(u + v) = mu + mv$
- 6) $(m + n)u = mu + nu$
- 7) $(mn)u = m(nu) = n(mu)$
- 8) $1u = u$
- 9) $0u = 0 = u0$
- 10) $\|cv\| = |c|\|v\|$

Example 5: Let $v = \langle -2, 5 \rangle$ and $w = \langle 3, 4 \rangle$, and find each of the following vectors.

a. $2v$

$v = \langle -2, 5 \rangle$, you have

$$\begin{aligned} 2v &= 2\langle -2, 5 \rangle \\ &= \langle 2(-2), 2(5) \rangle \\ &= \langle -4, 10 \rangle \end{aligned}$$

b. $w - v$

The difference of w and v is $w - v = \langle 3 - (-2), 4 - 5 \rangle = \langle 5, -1 \rangle$

c. $v + 2w$

$$\begin{aligned} &= \langle -2, 5 \rangle + \langle 2(3), 2(4) \rangle \\ &= \langle -2, 5 \rangle + \langle 6, 8 \rangle \\ &= \langle -2 + 6, 5 + 8 \rangle \\ &= \langle 4, 13 \rangle \end{aligned}$$