Using Fundamental Identities

We studied before the basic definitions and some properties of the individual trigonometric functions. In this lesson,we will use the fundamental identities to do the following:

1) Evaluate trigonometric functions
2) Simplify trigonometric expressions
3) Develop additional trigonometric identities

Let us derive the identities and we will start from the basic definitions that we learned earlier.

sin θ and cos θ are ratios defined as:

$\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$

Now, we use these results to find an important definition for tan θ:

$$\tan\theta = \frac{y}{x}$$

$$\tan\theta = \frac{y \div r}{x \div r}$$

$$\Rightarrow \tan\theta = \frac{\frac{y}{r}}{\frac{x}{r}}$$

$$But \sin\theta = \frac{y}{r}.......and.....\cos\theta = \frac{x}{r}$$

$$\Rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Identity 1: $\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$

Similarly we can deduce that $\cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta}$, $\sin\theta = \frac{1}{\csc\theta}$, $\cos\theta = \frac{1}{\sec\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\tan\theta = \frac{1}{\cot\theta}$, $\csc\theta = \frac{1}{\sin\theta}$,and $\cot\theta = \frac{1}{\tan\theta}$

Using the Pythagorean Theorem we obtain: $r^2 = x^2 + y^2$, dividing through by r^2 gives us:

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\Rightarrow \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$But \ \sin\theta = \frac{y}{r} \ .... \ and \ ..... \ \cos\theta = \frac{x}{r}$$

$$\Rightarrow (\sin\theta)^2 + (\cos\theta)^2 = 1$$

$$\therefore \sin^2\theta + \cos^2\theta = 1$$

Identity 2: $\sin^2\theta+\cos^2\theta=1$

We now proceed to derive two other related formulas that can be used when proving trigonometric identities.

It is suggested that you remember how to find the identities, rather than try to memorize each one of them!!!!

Dividing $\sin^2\theta+\cos^2\theta=1$ through by $\cos^2\theta$ gives us:

$$\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$
$$\Rightarrow\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$
$$\Rightarrow\left(\frac{\sin\theta}{\cos\theta}\right)^2+\frac{\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}}=\frac{1}{\cos^2\theta}$$
$$\Rightarrow(\tan\theta)^2+1=\left(\frac{1}{\cos\theta}\right)^2$$
$$\therefore\tan^2\theta+1=\sec^2\theta$$

Identity 3: $\tan^2\theta+1=\sec^2\theta$

Dividing $\sin^2\theta+\cos^2\theta=1$ through by $\sin^2\theta$ gives us:

$$\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$
$$\Rightarrow\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$
$$\Rightarrow\frac{\cos^2\theta}{\sin^2\theta}+\frac{\sin^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$
$$\Rightarrow\left(\frac{\cos\theta}{\sin\theta}\right)^2+\frac{\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}}=\frac{1}{\sin^2\theta}$$
$$\Rightarrow(\cot\theta)^2+1=\left(\frac{1}{\sin\theta}\right)^2$$
$$\therefore\cot^2\theta+1=\csc^2\theta$$

Identity 4: $\cot^2\theta+1=\csc^2\theta$

Fundamental Trigonometric Identities:

Reciprocal Identities:

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

Quotient Identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Example 1: Use the basic trigonometric identities to determine the other five values of the trigonometric functions given that $\sin\alpha = \frac{7}{8}$ and $\cos\alpha < 0$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\Rightarrow \left(\frac{7}{8}\right)^2 + \cos^2\alpha = 1$$

$$\Rightarrow \cos^2\alpha = 1 - \left(\frac{7}{8}\right)^2$$

$$\Rightarrow \cos^2\alpha = 1 - \frac{49}{64} = \frac{15}{64}$$

$$\cos\alpha = \pm\sqrt{\frac{15}{64}} \quad \text{but } \cos\alpha < 0$$

$$\Rightarrow \cos\alpha = -\sqrt{\frac{15}{64}}$$

$$\sin\alpha = \frac{7}{8} \quad \text{and} \quad \cos\alpha = -\sqrt{\frac{15}{64}} = \frac{-\sqrt{15}}{8}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{7/8}{-\sqrt{15}/8} = \frac{-7}{\sqrt{15}} = \frac{-7\sqrt{15}}{15}$$

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{1}{-7/\sqrt{15}} = \frac{-\sqrt{15}}{7}$$

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{8}{7}$$

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{1}{-\sqrt{15}/8} = -\frac{8}{\sqrt{15}} = -\frac{8\sqrt{15}}{15}$$

Example 2: If $\sin\theta = \frac{-5}{13}$, $\frac{3\pi}{2} < \theta < 2\pi$, find $\csc\theta$, $\tan\theta$, $\cot\theta$ and $\frac{\sin\theta - \csc\theta}{\tan\theta + \cot\theta}$

We know that $\frac{3\pi}{2} < \theta < 2\pi$ i.e, ($270^0 < \theta < 360^0$) i.e. θ lies in the 4^{th} Quadrant.

$\Rightarrow \sin\theta$, $\csc\theta$, $\tan\theta$ and $\cot\theta$ are negatives and $\cos\theta$ and $\sec\theta$ are positive.

$$\csc\theta = \frac{1}{\sin\theta} \Rightarrow \csc\theta = -\frac{13}{5}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow \left(\frac{-5}{13}\right)^2 + \cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = 1 - \frac{25}{169} = \frac{144}{169} \qquad \therefore \cos\theta = \frac{12}{13}$$

$$\sin\theta = \frac{-5}{13} \quad \text{and} \quad \cos\theta = \frac{12}{13}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-5/13}{12/13} = \frac{-5}{12}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-5/12} = -\frac{12}{5}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{12/13} = \frac{13}{12}$$

$$\frac{\sin\theta - \csc\theta}{\tan\theta + \cot\theta} = \frac{-5/13 - \left(-13/5\right)}{-5/12 + \left(-12/5\right)} = -\frac{1728}{2197}$$

Example 3: If cot α = 3 and $\pi < \alpha < \frac{3\pi}{2}$. Find $4\sin\alpha + 2\cos\alpha$

$\pi < \alpha < \frac{3\pi}{2}$ means a lies in 3^{rd} Quadrant.

$\therefore \sin\alpha$ and $\cos\alpha$ are negatives.

$$\csc^2\alpha = 1 + \cot^2\alpha = 1 + 3^2 = 10$$

$$\therefore \csc\alpha = -\sqrt{10}$$

$$\therefore \sin\alpha = \frac{1}{\csc\alpha} = \frac{1}{-\sqrt{10}}$$

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} \Rightarrow \cos\alpha = \sin\alpha\cot\alpha = 3\left(\frac{1}{-\sqrt{10}}\right) = \frac{-3}{\sqrt{10}}$$

$$\therefore 4\sin\alpha + 2\cos\alpha = 4\left(\frac{1}{-\sqrt{10}}\right) + 2\left(\frac{-3}{\sqrt{10}}\right) = \frac{-10}{\sqrt{10}} = -\sqrt{10}$$

Example 4: If $\frac{\sin A}{3} = \frac{\sin B}{4} = \frac{1}{5}$ where A and B are angles in the second Quadrant.

Prove that: $4\cos A + 3\cos B = -5$

1) We have $\frac{\sin A}{3} = \frac{1}{5} \Rightarrow \sin A = \frac{3}{5}$

$\Rightarrow$ A is Quadrant II.

$$\sin A = \frac{3}{5}$$
$$\sin^2 A + \cos^2 A = 1$$
$$\Rightarrow \left(\frac{3}{5}\right)^2 + \cos^2 A = 1$$
$$\Rightarrow \cos^2 A = 1 - \frac{9}{25} = \frac{16}{25} \qquad \therefore \cos A = -\frac{4}{5}$$

) We have $\frac{\sin B}{4} = \frac{1}{5} \Rightarrow \sin B = \frac{4}{5}$

$\Rightarrow$ B is in Quadrant II.

$$\sin B = \frac{4}{5}$$
$$\sin^2 B + \cos^2 B = 1$$
$$\Rightarrow \left(\frac{4}{5}\right)^2 + \cos^2 B = 1$$
$$\Rightarrow \cos^2 B = 1 - \frac{16}{25} = \frac{9}{25} \qquad \therefore \cos B = -\frac{3}{5}$$

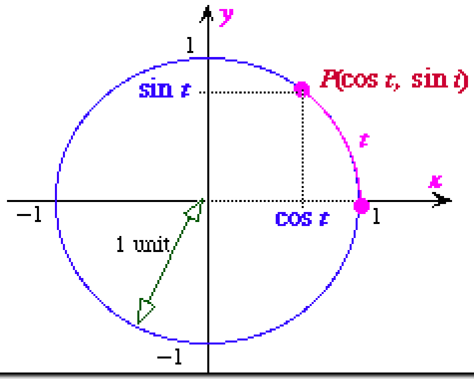
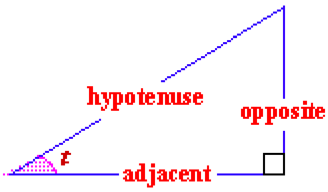
$$4\cos A + 3\cos B = 4\left(-\frac{4}{5}\right) + 3\left(-\frac{3}{5}\right) = -\frac{16}{5} - \frac{9}{5} = -\frac{25}{5} = -5$$

Example 5: If $7\cos\theta - 24\sin\theta = 0$ and θ being in the 1^{st} Quadrant. Find the values of $\tan\theta$, and sec θ .

$$7\cos\theta - 24\sin\theta = 0$$
$$\Rightarrow 7\cos\theta = 24\sin\theta$$
$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{7}{24}$$
$$\Rightarrow \tan\theta = \frac{7}{24}$$

$$\sec^2\theta = \tan^2\theta + 1$$
$$\Rightarrow \sec^2\theta = \left(\frac{7}{24}\right)^2 + 1$$
$$\Rightarrow \sec^2\theta = \frac{625}{576}$$
$$\Rightarrow \sec\theta = \frac{25}{24}$$

The Trigonometric Functions as Ratios in a Right Triangle

Defining Formula	Ratio in Right Triangle
	
sin t = y-coordinate of point P	$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$
cos t = x-coordinate of point P	$\cos\theta = \frac{Adjacent}{\text{Hypotenuse}}$
$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\tan\theta = \frac{\text{Opposite}}{Adjacent}$
$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$\cot\theta = \frac{Adjacent}{\text{Opposite}}$
$\sec\theta = \frac{1}{\cos\theta}$	$\sec\theta = \frac{Hypotenuse}{Adjacent}$
$\csc\theta = \frac{1}{\sin\theta}$	$\csc\theta = \frac{Hypotenuse}{\text{Opposite}}$