Trigonometry Functions of Any Angle

The definition of trigonometric functions were restricted to acute angles . Here, the definition will be extended to cover any angle.

Definition 1: Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and

$r=\sqrt{x^2+y^2}\neq 0$

$\sin\theta=\dfrac{y}{r}$

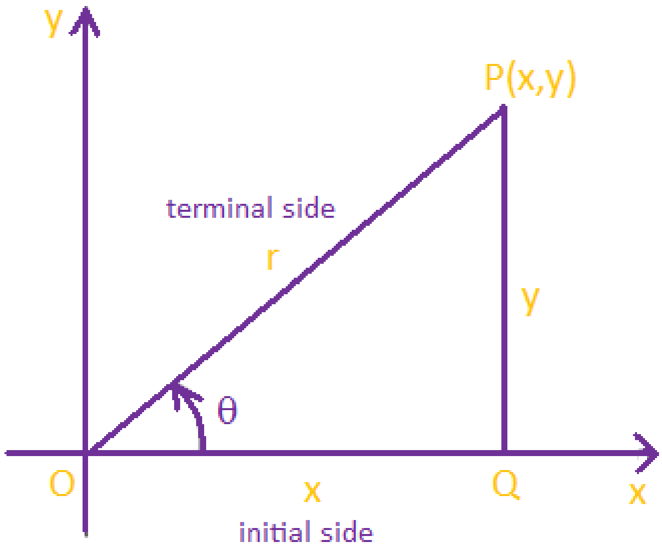
$\cos\theta=\dfrac{x}{r}$

$\tan\theta=\dfrac{y}{x};x\neq 0$

$\cot\theta=\dfrac{x}{y};y\neq 0$

$\sec\theta=\dfrac{r}{x};x\neq 0$

$\csc\theta=\dfrac{r}{y};y\neq 0$



Because $r=\sqrt{x^2+y^2}$ cannot be zero, it follows that the sine and cosine functions are defined for any real value of θ. However, if $x=0$, the tangent and secant of θ are undefined. For example the tangent of 90° is undefined. Similarly, if $y=0$, the cotangent and cosecant of θ are undefined.

Note: To find *r*, we are actually using the Pythagorean Theorem, since we have a right angled triangle.

Example 1: Let (− 3, 4) be a point on the terminal side of θ.Find the values of the six trigonometric functions of θ

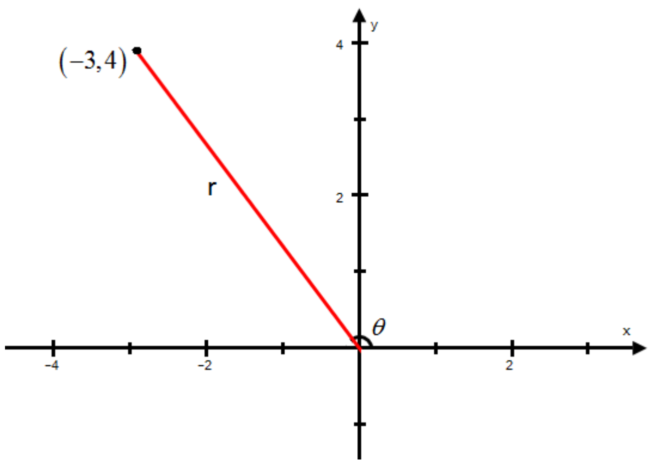
$r=\sqrt{x^2+y^2}=\sqrt{(-3)^2+4^2}$

$\Rightarrow r=\sqrt{9+16}=\sqrt{25}=5$

$\sin\theta=\dfrac{y}{r}=\dfrac{4}{5};\cos\theta=\dfrac{x}{r}=-\dfrac{3}{5}$

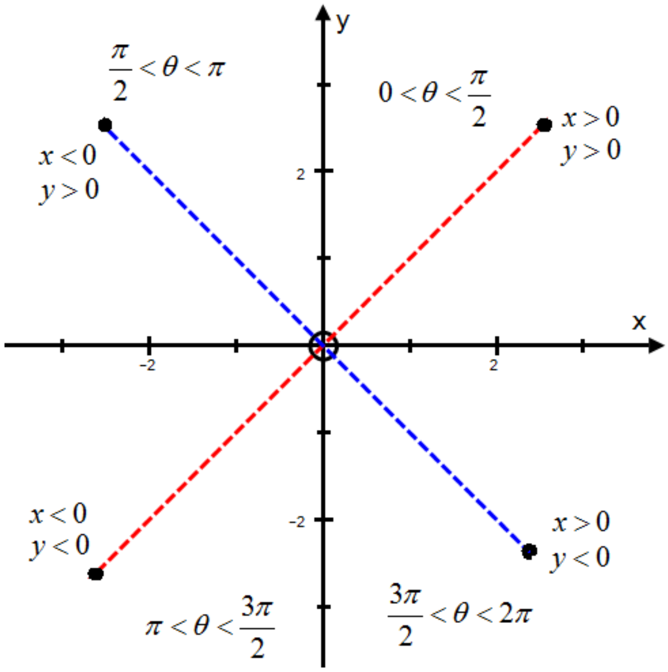
$\tan\theta=\dfrac{y}{x}=-\dfrac{4}{3};\cot\theta=\dfrac{x}{y}=-\dfrac{3}{4}$

$\sec\theta=\dfrac{r}{x}=-\dfrac{5}{3};\csc\theta=\dfrac{r}{y}=-\dfrac{5}{4}$



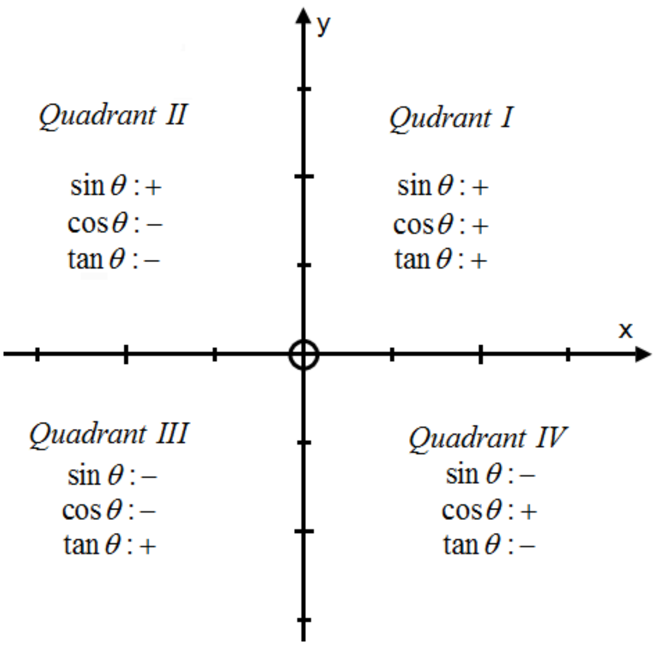
Signs of Function Values

We can tell the interval of the angle θ from the signs of the coordinates of the given point. In the previous example, the point (-3,4) is in the second quadrant so the value of the angle θ is between 90° and 180°. Also, we can deduce the signs of the trigonometric functions. The diagram shows the interval of the angle and the relationship with the signs of the x-values and y-values.



The sign of r is positive all the time because the square root of any number is positive, so, the sign of the sine function is the same as the sign of the y coordinate of the point and the sign of the cosine function is the same as the sign of the x-coordinate of the point.

Therefore, in quadrants I and IV $\sin\theta > 0$ and in quadrants II and III $\sin\theta < 0$. Similarly, we can deuce the signs of the other trigonometric functions. In the diagram the signs of $\sin\theta; \cos\theta; \tan\theta$ are shown in each quadrant.



What is a quick way of finding the signs of trig values for angles in a particular Quadrant?
"**ASTC**"
Think: "**All Students Take Calculus**". Start in Quadrant I and progress counterclockwise through the Quadrants:

S	**A**
T	**C**

- **A**ll of the six basic trig functions are positive in Quadrant I.
  (They are all positive for acute angles.)

- **S**in and its reciprocal, Csc, are positive in Quadrant II.
  (The other four functions are negative.)

- **T**an and its reciprocal, Cot, are positive in Quadrant III.

- **C**os and its reciprocal, Sec, are positive in Quadrant IV.

<u>Reference Angles</u>

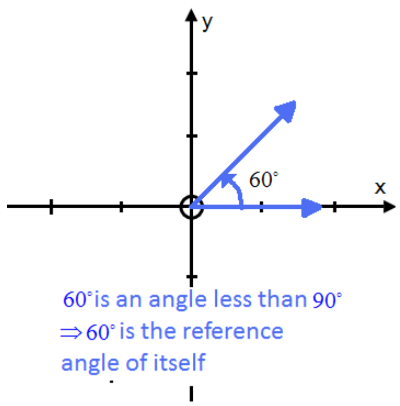
The values of the trigonometric functions of angles greater than 90° or less than 0° can be determined from their values at corresponding acute angles called **reference angles.**

Definition 2: Let θ be an angle in standard position. Its reference angle is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Assume angle A is positive and less than 360° (2π), we have 4 possible cases:

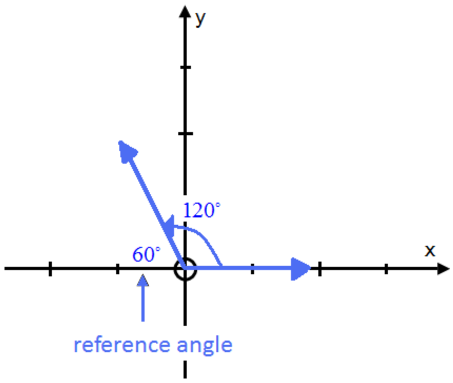
1. If angle A is in **quadrant I** then the reference angle.

$A_r = A$



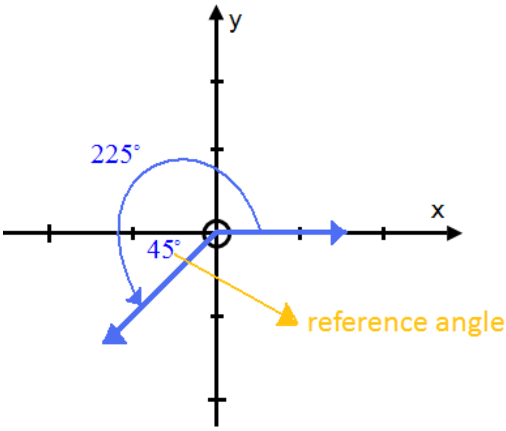
2. If angle A is in **quadrant II** then the reference angle

$\mathbf{A_r = 180^{\circ} - A}$ if A is given degrees
and
$\mathbf{A_r = \pi - A}$ if A is given in radians.



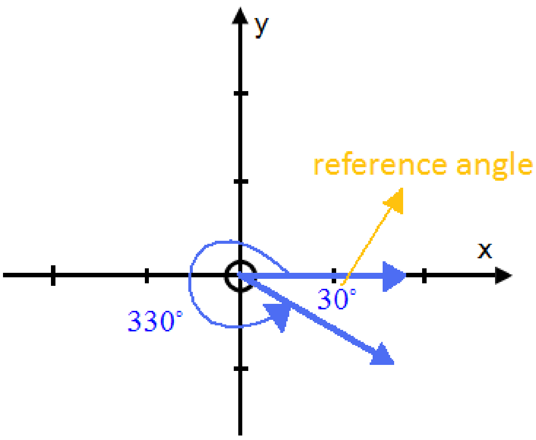
3. If angle A is in **quadrant III** then the reference angle

$\mathbf{A_r = A - 180^{\circ}}$ if A is given degrees
and
$\mathbf{A_r = A - \pi}$ if A is given in radians.



4. If angle A is in **quadrant IV** then the reference angle

$\mathbf{A_r = 360^{\circ} - A}$ if A is given degrees
and
$\mathbf{A_r = 2\pi - A}$ if A is given in radians.



Example 2: Find the reference angle

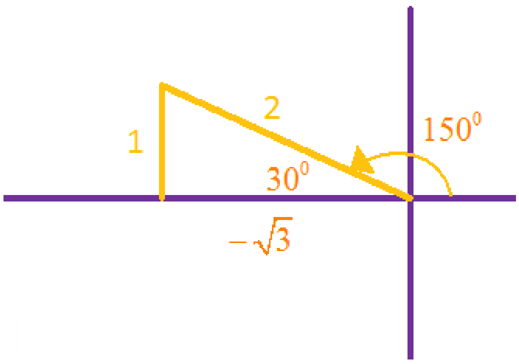
1) A = 120 °.
   Angle A is in quadrant II and the reference angle is given by
   A_r = 180° - 120° = 60°

2) A = $-\dfrac{15\pi}{4}$.
   The given angle is not positive and less than 2π. We can use the positive and less than 2π co terminal A_c to angle A.
   $A_c = -\dfrac{15\pi}{4}+2(2\pi)=\dfrac{\pi}{4}$
   Angle A and A_c are co terminal and have the same reference angle. A_c is in quadrant I, therefore
   $A_r = A_c = \dfrac{\pi}{4}$

3) A = - 30 °
   Angle A is negative, in quadrant IV and its absolute value is less than 90 °. Hence
   A_r = | -30 ° | = 30 °

Example 3: Find the values of the six trigonometric functions for θ = 150 ^{0}

Step 1: Draw the angle in standard position
Step 2: Draw the reference triangle and angle
Step 3: Label the triangle. Here we will label using the standard $30^0 - 60^0 - 90^0$ triangle



NOTE: Since one side of the reference triangle is on the negative x-axis that side is labeled as $-\sqrt{3}$. This is VERY IMPORTANT. You will notice that this makes the cosine, secant, tangent and cotangent negative.

Now we can find the 6 trigonometric functions by reading them off the reference triangle:

$$\sin(150^0) = \frac{opp}{hyp} = \frac{1}{2} \quad ; \quad \cos(150^0) = \frac{adj}{hyp} = -\frac{\sqrt{3}}{2}$$

$$\tan(150^0) = \frac{opp}{adj} = -\frac{1}{\sqrt{3}} \quad ; \quad \cot(150^0) = \frac{adj}{opp} = -\frac{\sqrt{3}}{1} = -\sqrt{3}$$

$$\sec(150^0) = \frac{hyp}{adj} = -\frac{2}{\sqrt{3}} \quad ; \quad \csc(150^0) = \frac{hyp}{opp} = \frac{2}{1} = 2$$

From this example we can conclude a very important rule.

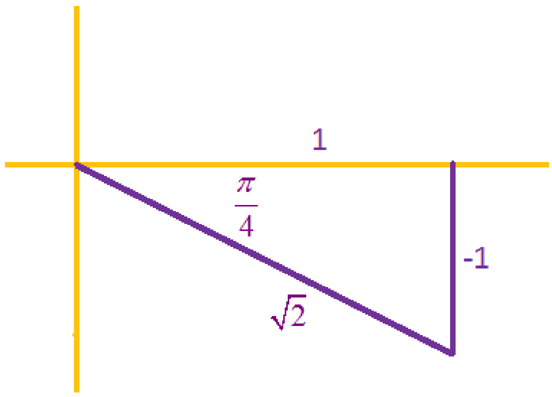
Rule 1: Using the definitions of the six trigonometric functions we can determine the values for $(\pi - \theta)$ angles in terms of values of θ angles and in summary we have:

$$\sin(\pi - \theta) = \sin\theta \qquad \cos(\pi - \theta) = -\cos\theta \qquad \tan(\pi - \theta) = -\tan\theta$$

$$\cot(\pi - \theta) = -\cot\theta \qquad \csc(\pi - \theta) = \csc\theta \qquad \sec(\pi - \theta) = -\sec\theta$$

Example 4: Find the values of the six trigonometric functions for $-\frac{\pi}{4}$

Note that the angle is negative.
Step 1: Draw the angle in standard position
Step 2: Draw the reference triangle and angle
Step 3: Label the triangle. Here we will label using the standard 30 – 60 - 90 triangle



NOTE: Since one side of the reference triangle is in the negative y direction that side is labeled as -1. This is VERY IMPORTANT. You will notice that this makes the sine, cosecant, tangent and cotangent negative.

Now we can find the 6 trigonometric functions by reading them off the reference triangle:

$$\sin\left(-\frac{\pi}{4}\right)=\frac{opp}{hyp}=-\frac{1}{\sqrt{2}} \quad ; \quad \cos\left(-\frac{\pi}{4}\right)=\frac{adj}{hyp}=\frac{1}{\sqrt{2}}$$

$$\tan\left(-\frac{\pi}{4}\right)=\frac{opp}{adj}=-\frac{1}{1}=-1 \quad ; \quad \cot\left(-\frac{\pi}{4}\right)=\frac{adj}{opp}=-\frac{1}{1}=-1$$

$$\sec\left(-\frac{\pi}{4}\right)=\frac{hyp}{adj}=\sqrt{2} \quad ; \quad \csc\left(-\frac{\pi}{4}\right)=\frac{hyp}{opp}=-\frac{\sqrt{2}}{1}=-\sqrt{2}$$

Rule 2: (Formulas for Negatives) using the definitions of the six trigonometric functions we can determine the values for negative angles in terms of values of positive angles and in summary we have:

$$\sin(-\theta)=-\sin\theta \qquad \cos(-\theta)=\cos\theta \qquad \tan(-\theta)=-\tan\theta$$

$$\cot(-\theta)=-\cot\theta \qquad \csc(-\theta)=-\csc\theta \qquad \sec(-\theta)=\sec\theta$$

Rule 3: Using the definitions of the six trigonometric functions we can determine the values for $(\pi+\theta)$ angles in terms of values of positive θ angles and in summary we have:

$$\sin(\pi+\theta)=-\sin\theta \qquad \cos(\pi+\theta)=-\cos\theta \qquad \tan(\pi+\theta)=\tan\theta$$

$$\cot(\pi+\theta)=\cot\theta \qquad \csc(\pi+\theta)=-\csc\theta \qquad \sec(\pi+\theta)=-\sec\theta$$

Rule 4: Identities expressing trig functions in terms of their complements:

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \qquad \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$$

$$\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta \qquad \csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta \qquad \sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta$$

Definition 3: (**Periodic Function**) A function f is periodic if there exists a positive real number k such that $f(\theta+k)=f(\theta)$ for every θ in the domain of f .The least positive real number k ,if it exists, is the period of f.

Proposition 1: (**Periodic Function**) The six trigonometric functions are periodic functions. Sine, cosine, secant, and cosecant have period 2π while tangent and cotangent have period π.

$$\sin(\theta + 2\pi) = \sin\theta$$

$$\cos(\theta + 2\pi) = \cos\theta$$

$$\tan(\theta + \pi) = \tan\theta$$

Definition 4: (**Even Function**) A function of a real number x is called an even function when $f(-x) = f(x)$ for all x in the domain of f

Definition 5: (**Odd Function**) A function of a real number x is called an odd function when $f(-x) = -f(x)$ for all x in the domain of f

Proposition 2: (Parity of the Trigonometric Functions) The cosine and secant functions are even functions and the sine, cosecant, tangent and cotangent functions are odd functions.

Reference Angle Values in Various Quadrants

Function	**Quadrant**		
	II	III	IV
sin θ	sin (180° − θ)	−sin (θ + 180 °)	−sin (360° − θ)
cos θ	−cos (180° − θ)	−cos (θ + 180°)	cos (360° − θ)
tan θ	−tan (180° − θ)	tan (θ + 180°)	−tan (360° − θ)
cot θ	−cot (180° − θ)	cot (θ + 180°)	−cot (360° − θ)
sec θ	−sec (180° − θ)	-sec (θ + 180°)	sec (360° − θ)
csc θ	csc (180° − θ)	-csc (θ + 180°)	−csc (360° − θ)