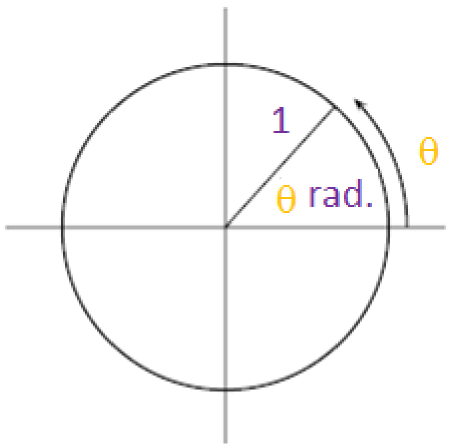
Trigonometric Functions: The Unit Circle

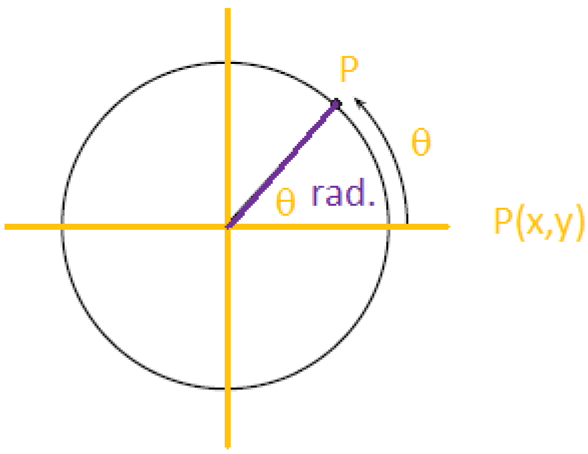
In the 1600s, scientists began using trigonometry to solve problems in physics and engineering. Such applications necessitated extending the domains of the trigonometric functions to include all real numbers, not just a set of angles. This extension was accomplished by using a correspondence between an angle and the length of an arc on a **unit circle.**

The two historical perspectives of trigonometry incorporate different methods for introducing the trigonometric functions. Our first introduction to these functions is based on the unit circle.

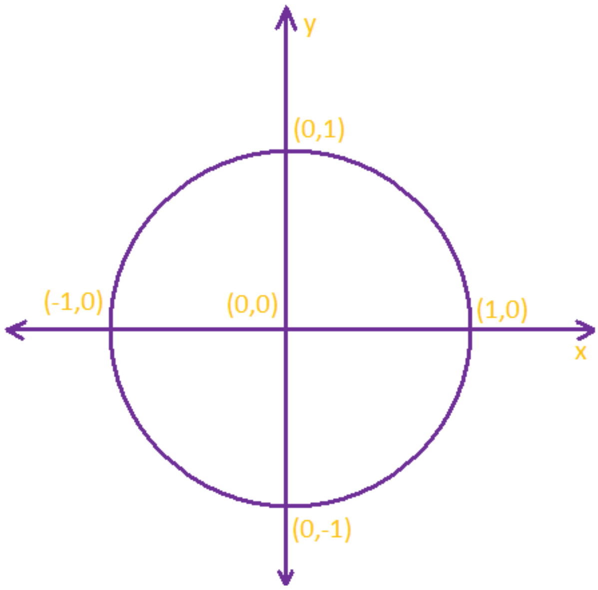
Consider a unit circle (radius = 1) centered at the origin (O(0,0)). Recall the formula $s = r\theta$. This relates the length of an arc of a circle with the radius of the circle and the central angle θ (in radians). For the unit circle, $r = 1 \Rightarrow s = \theta$, that is, the arc length subtended ("marked off") by an angle is equal to the angle (in radians).



Let (x, y) be the coordinates of a point P on the unit circle as it moves in a counterclockwise direction from the positive x-axis. The point P also determines an angle θ (in radians) which, as mentioned above, is equal to the length of the arc traveled by P.



Consider the unit circle given by: $x^2 + y^2 = 1$



Each point on the outside of the unit circle corresponds to a specific value of central angle θ in standard position. The real number 0 corresponds to the point (1,0). Moreover, because the unit circle has a circumference of 2π, the real number 2π also corresponds to the point (1,0).

Each real number t corresponds to a central angle θ (in standard position) whose radian measure is t. With this interpretation of t, the arc length formula $s = r\theta$ (with r=1) indicates that the real number t is the length of the arc intercepted by the angle θ, given in radians.

The coordinates x and y are two functions of the real variable t. You can use these coordinates to define the six trigonometric functions of t; sine, cosine, secant, cosecant, tangent, cotangent. These six functions are normally abbreviated sin, csc, cos, sec, tan and cot respectively.

Definition 1: Let t be a real number and let (x, y) be the point on the **unit circle** corresponding to t.

$\sin t = y$ $\qquad$ $\cos t = x$ $\qquad$ $\tan t = \frac{y}{x}, x \neq 0$

$\csc t = \frac{1}{y}, y \neq 0$ $\qquad$ $\sec t = \frac{1}{x}, x \neq 0$ $\qquad$ $\cot t = \frac{x}{y}, y \neq 0$

Definition 1 can be extended to all circles not only the unit circle. If a circle has a radius of length r, $x^2 + y^2 = r^2$ then we have:

$\sin t = \frac{y}{r}$ $\qquad$ $\cos t = \frac{x}{r}$ $\qquad$ $\tan t = \frac{y}{x}, x \neq 0$

$\csc t = \frac{r}{y}, y \neq 0$ $\qquad$ $\sec t = \frac{r}{x}, x \neq 0$ $\qquad$ $\cot t = \frac{x}{y}, y \neq 0$

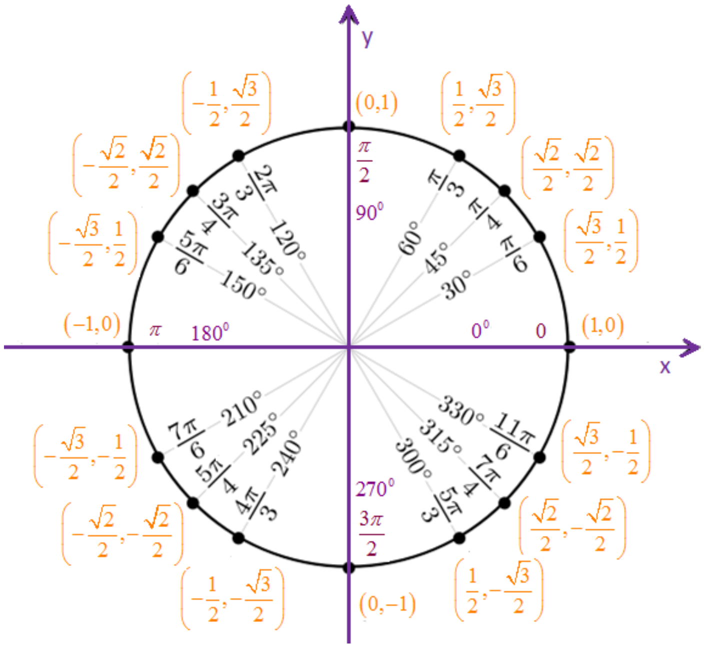
From the definition we note that the tangent and secant functions are not defined when x=0.

Because $t=\dfrac{\pi}{2}$ corresponds to $(x,y)=(0,1)$, it follows that $\tan\left(\dfrac{\pi}{2}\right)$ and $\sec\left(\dfrac{\pi}{2}\right)$ are undefined.

Similarly cotangent and cosecant are not defined when y=0. For instance, because t=0 corresponds to $(x,y)=(1,0)$, $\cos 0$ and $\csc 0$ are undefined.

The unit circle below is divided into 16 arcs corresponding to t-values of:

$0,\ \dfrac{\pi}{6},\ \dfrac{\pi}{4},\ \dfrac{\pi}{3},\ \dfrac{\pi}{2},\ \dfrac{2\pi}{3},\ \dfrac{3\pi}{4},\ \dfrac{5\pi}{6},\ \pi,\ \dfrac{7\pi}{6},\ \dfrac{5\pi}{4},\ \dfrac{4\pi}{3},\ \dfrac{3\pi}{2},\ \dfrac{5\pi}{3},\ \dfrac{7\pi}{4},\ \dfrac{11\pi}{6}\ and\ 2\pi$



To verify the points on the unit circle, note that $\left(\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)$ lies on the line y=x and on the unit circle

$$x^2+y^2=1.$$
$$\Rightarrow x^2+x^2=1$$
$$\Rightarrow 2x^2=1$$
$$\Rightarrow x^2=\frac{1}{2}$$
$$\Rightarrow x=\pm\frac{\sqrt{2}}{2}$$

Substituting in the equation y=x

$$x=\frac{\sqrt{2}}{2}\Rightarrow y=\frac{\sqrt{2}}{2}\Rightarrow\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$$
$$x=-\frac{\sqrt{2}}{2}\Rightarrow y=-\frac{\sqrt{2}}{2}\Rightarrow\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$$

Example 1: Evaluate the six trigonometric functions for:

1) $t=\frac{\pi}{6}$
2) $t=\frac{5\pi}{4}$

We begin by finding the corresponding point (x,y) on the unit circle, then we use the definition of the trigonometric functions.

1) $t=\frac{\pi}{6}$

$t=\frac{\pi}{6}$ corresponds to the point $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

$$\sin\frac{\pi}{6}=y=\frac{1}{2} \quad ; \quad \cos\frac{\pi}{6}=x=\frac{\sqrt{3}}{2}$$
$$\tan\frac{\pi}{6}=\frac{y}{x}=\frac{\sqrt{3}}{3} \quad ; \quad \cot\frac{\pi}{6}=\frac{x}{y}=\sqrt{3}$$
$$\csc\frac{\pi}{6}=\frac{1}{y}=2 \quad ; \quad \sec\frac{\pi}{6}=\frac{1}{x}=\frac{2\sqrt{3}}{3}$$

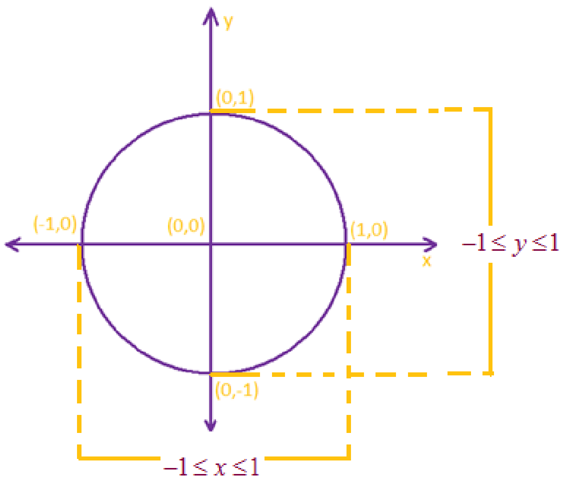
2) $t=\frac{5\pi}{4}$

$t=\frac{5\pi}{4}$ corresponds to the point $\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$

$$\sin\frac{5\pi}{4}=y=-\frac{\sqrt{2}}{2} \quad ; \quad \cos\frac{5\pi}{4}=x=-\frac{\sqrt{2}}{2}$$
$$\tan\frac{5\pi}{4}=\frac{y}{x}=1 \quad ; \quad \cot\frac{5\pi}{4}=\frac{x}{y}=1$$
$$\csc\frac{5\pi}{4}=\frac{1}{y}=-\sqrt{2} \quad ; \quad \sec\frac{5\pi}{4}=\frac{1}{x}=-\sqrt{2}$$

Definition 2: The domain of the sine and cosine functions is the set of all real numbers.

To determine the range of these two functions, consider the unit circle below:



(x, y) is on the unit circle $\Rightarrow -1 \le x \le 1$ and $-1 \le y \le 1$; By definition, $\sin t = y$ and $\cos t = x$ so the values of sint and cost also range between -1 and 1.

Definition 3: The range of the sine and cosine functions is:

$-1 \le y \le 1 \Rightarrow -1 \le \sin t \le 1$; $-1 \le x \le 1 \Rightarrow -1 \le \cos t \le 1$

Example 2: The given point *P* is located on the unit circle. Find the values of the six trigonometric functions

$$P\left(\frac{20}{29}, -\frac{21}{29}\right)$$

$$x = \frac{20}{29} \quad ; \quad y = -\frac{21}{29}$$

$$\sin\theta = y = -\frac{21}{29} \quad ; \quad \cos\theta = x = \frac{20}{29}$$

$$\tan\theta = \frac{y}{x} = \frac{-21}{20} \quad ; \quad \cot\theta = \frac{x}{y} = \frac{20}{-21}$$

$$\csc\theta = \frac{1}{y} = \frac{-29}{21} \quad ; \quad \sec\theta = \frac{1}{x} = \frac{29}{20}$$