Solving Polynomial and Rational Inequalities

Put the inequality in standard form.
To do this collect all terms on one side.
If you have fractions obtain a single fraction by adding / subtracting all fractions.
Do not clear any denominators that contain variables.

Standard form.
For polynomials: $P(x)\ \le\ ,\ <\ ,\ \ge\ ,\ >\ 0$

For rational expressions (fractions): $\dfrac{p(x)}{q(x)}\ \le\ ,\ <\ ,\ \ge\ ,\ >\ 0$

Polynomial case: Left side is a polynomial $P(x)$

(1) Find all zeros of $P(x)$ by solving the equation $P(x)=0$
These are called the **critical numbers**.

(2) Place the critical numbers found on a real number line. These numbers will split the real number line into a number of intervals.

(3) Make a "table of signs"
To do this pick test numbers (other than the critical numbers) in each interval found above and find the sign of $P(x)$ in that interval by evaluating $P(x)$ at the corresponding test number. Record that sign on the table.

Note: The sign of $P(x)$ is the same throughout the interval so it is enough to check the sign at a single test number.

(4) Obtain the solution, check endpoints.
Collect the intervals for which the sign found as above is as desired.
If the inequality includes the equal sign (non-strict inequality) then you should include the endpoints of the intervals.

Rational case: Left side is a rational expression $\dfrac{p(x)}{q(x)}$ **in lowest terms.**

(Numerator and denominator are polynomials)
(1a) Find the zeros of the numerator by solving the equation $p(x)=0$
These are the numbers where the fraction itself is **<u>zero</u>**.
(1b) Find the zeros of the denominator by solving the equation $q(x)=0$
These are the numbers where the fraction itself is **<u>undefined</u>**.
All zeros (of numerator and denominator) are called the **critical numbers**.

(2) and (3) Follow the same steps as in (2), (3) of part 2. This time, place on the number line all the critical numbers.

You test the sign of the fraction $\frac{p(x)}{q(x)}$ for each of the intervals.

(4) Obtain the solution.

Collect the intervals where the sign of $\frac{p(x)}{q(x)}$ is as desired.

(4a) If the inequality is strict (i.e. no equality is included, like $<$, $>$) then you do not include the endpoints.

(4b) If the inequality is non-strict (i.e. equality is included, like $\leq$, $\geq$) then you should include the endpoints of the intervals for which the denominator is non-zero.

Note: The endpoints of the intervals where the denominator is zero have to be excluded from the solution always since at those points the rational expression is undefined.

Example 1: Solve the inequality $x^3+x^2\leq 2x$

Put in standard form and factor,

$$x^3+x^2\leq 2x \quad\Rightarrow\quad x^3+x^2-2x\leq 0 \quad\Rightarrow\quad x(x+2)(x-1)\leq 0$$

Thus, equivalently, we need to solve the inequality, $x(x+2)(x-1)\leq 0$

Find the zeros (critical numbers), $x(x+2)(x-1)=0 \quad\Rightarrow\quad x=0 \text{ and } x=-2 \text{ and } x=1$

Place the critical numbers on the real number line, pick test numbers and make the table of signs,

Test numbers		-3		-1		1/2		2	
Critical numbers	$-\infty$		-2		0		1		∞
$x(x+2)(x-1)$		(-)(-)(-)		(-)(-)(+)		(+)(-)(+)		(+)(+)(+)	
overall sign		-		+		-		+	

We need to pick the intervals with the – sign since we solve, $x(x+2)(x-1)\leq 0$

Note that the ends points are included (solid circles) since the inequality contains the equal sign.

The solution is $(-\infty,-2]\cup[0,1]$.

Example 2: Solve the inequality, $x^3>3x^2$

$$x^3>3x^2 \quad\Rightarrow\quad x^3-3x^2>0 \quad\Rightarrow\quad x^2(x-3)>0$$

Thus, equivalently, we need to solve the inequality, $x^2(x-3)>0$

Find the zeros (critical numbers), $x^2(x-3)=0 \quad\Rightarrow\quad x=0 \text{ and } x=3$

Place the critical numbers on the real number line, pick test numbers and make the table of signs,

Test numbers		-1		1		4	
Critical numbers	$-\infty$		0		3		∞
$x^2(x-3)$		(+)(-)		(+)(-)		(+)(+)	
overall sign		-		-		+	

We need to pick the interval with the + sign since we solve $x^2(x-3)>0$
Note that the ends points are not included (open circles) since the inequality does not contain the equal sign. The solution is then, $(3,\infty)$.

Remark: This example can be shortened a lot if we note the presence of x^2 in the inequality,
$x^2(x-3)>0$
Observe that the sign of x^2 is always non-negative (since it is a squared term) thus the sign of the expression $x^2(x-3)$ is the same as the sign of $x-3$.
Since we want $x^2(x-3)>0$ it is enough to have, $x-3>0$ which gives the solution: $x>3$.

Example 3: Solve the inequality $\dfrac{x^2(x-1)}{x+2} \le 0$

This is a rational inequality and it is already in standard form.
To find the critical numbers find the zeros of the numerator and denominator:

Zeros of numerator: $x^2(x-1) = 0 \Rightarrow x=0$ *and* $x=1$

Zeros of the denominator: $x+2 = 0 \Rightarrow x=-2$

Place the critical numbers on the real number line, pick test numbers and make the table of signs,

Critical numbers: $-\infty$, -2, 0, 1, ∞

Test numbers	-3	-1	1/2	2
$\dfrac{x^2(x-1)}{x+2}$	$\dfrac{(+)(-)}{(-)}$	$\dfrac{(+)(-)}{(+)}$	$\dfrac{(+)(-)}{(+)}$	$\dfrac{(+)(+)}{(+)}$
overall sign	+	-	-	+

Note: Since the inequality includes the equal sign, we should include in the solution the zeros of the numerator 0, 1 (filled circles).
However we should exclude the zero of the denominator –2 (empty circle) since it makes the expression undefined. The solution is then $(-2,1]$.

Example 4: Solve the inequality $\dfrac{3}{x-1} \ge \dfrac{2}{x+1}$

This is a rational inequality but we have to put it in standard form. We should not clear denominators and we should get a single fraction on one side.

$$\frac{3}{x-1} \ge \frac{2}{x+1} \Rightarrow \frac{3}{x-1} - \frac{2}{x+1} \ge 0$$

Use the LCD $(x-1)(x+1)$ to subtract the fraction on the left to get a single fraction:

$$\frac{3(x+1)}{(x-1)(x+1)} - \frac{2(x-1)}{(x+1)(x-1)} \ge 0 \Rightarrow \frac{3x+3-2x+2}{(x-1)(x+1)} \ge 0 \Rightarrow \frac{x+5}{(x-1)(x+1)} \ge 0$$

Thus we need to solve the inequality, $\dfrac{x+5}{(x-1)(x+1)} \ge 0$

To find the critical numbers find the zeros of the numerator and denominator:
Zeros of the numerator: $x+5=0 \;\Rightarrow\; x=-5$
Zeros of the denominator: $(x-1)(x+1)=0 \;\Rightarrow\; x=-1 \text{ and } x=1$
Place the critical numbers on the real number line, pick test numbers and make the table of signs,

Critical numbers: $-\infty$, -5, -1, 1, ∞

Test numbers	-6	-2	0	2
$\frac{x+5}{(x-1)(x+1)}$	$\frac{(-)}{(-)(-)}$	$\frac{(+)}{(-)(-)}$	$\frac{(+)}{(-)(+)}$	$\frac{(+)}{(+)(+)}$
overall sign	-	+	-	+

We need to pick the interval with the + sign since we solve $\frac{x+5}{(x-1)(x+1)} \geq 0$

Note: Since the inequality includes the equal sign, we should include in the solution the zeros of the numerator −5 (filled circle).
However we should exclude the zeros of the denominator −1, 1 (empty circles) since they make the expression undefined. The solution is then, $[-5,-1)\cup(1,\infty)$.

Guidelines to solve quadratic inequalities:
Step 1: Put the inequality in standard form: $ax^2+bx+c \;\; >, \geq, <, \leq \; 0$
Step 2: Set the quadratic equal to zero and solve: $ax^2+bx+c=0$
Step 3: Find the zeros using factoring or the quadratic formula.

Rule 1: **In between the zeros the sign of the quadratic will be the opposite of the sign of *a*. Outside the zeros the sign of the quadratic will the same as the sign of *a*.**

Example 5: Solve the inequality, $3x^2+5x-2<0$

This is a quadratic inequality already in standard form.
Set the quadratic equal to zero and solve: $3x^2+5x-2=0$
Here we can factor, we get, $(3x-1)(x+2)=0 \;\Rightarrow\; 3x-1=0\;,\; x+2=0 \;\Rightarrow\; x=-2 \text{ and } x=\frac{1}{3}$
Since we solve $3x^2+5x-2<0$ we need to find the intervals where the quadratic is negative. We observe that $a=3>0$ (positive).
The quadratic will be negative in between the zeros since there, the sign is opposite of the sign of *a*.
Thus the solution is $(-2,1/3)$.
Note we do not include the endpoints since the inequality we solve does not include the equal sign.