Solving Linear Equations

Our aim is to find the values of the n unknowns $x_1, x_2, x_3, ..., x_n$ that simultaneously satisfy the system of m linear equations

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
.
.
.
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

where a_{ij} and b_i bi are constants. The values of $x_1, x_2, x_3, ..., x_n$ that satisfy the equations are called the solution. The equations are a "system" because there is more than one equation, and "linear" because the unknowns are not multiplied together (for example, there are no x_1x_2 or x_1^2 terms).

Rule 1: Gaussian elimination

To find the solution of these equations we use a method called Gaussian elimination. There are five steps to Gaussian elimination.

Represent the equations as an augmented matrix.

$$\begin{bmatrix} a_{11} & a_{12}\ldots\ldots\ldots a_{1n} & | \, b_1 \\ a_{21} & a_{22}\ldots\ldots\ldots a_{2n} & | \, b_2 \\ . & . \quad\quad\quad . & | \, . \\ . & . \quad\quad\quad . & | \, . \\ a_{m1} & a_{m2}\ldots\ldots\ldots a_{mn} & | \, b_n \end{bmatrix}$$

Row reduces the matrix into rank or reduced rank form.

Reform the equations from the form. The new system of equations has the same solution as the original system.

Solve by back substitution.

Check answer.

Gaussian elimination is an excellent method for solving simultaneous linear equations.

Example 1: Find the solution of the system of equations

x − 2y = −10
3x + y = 12

1. **Create the augmented matrix.**

$$\begin{bmatrix} 1 & -2 \,|\, -10 \\ 3 & 1 \,|\, \;\; 12 \end{bmatrix}$$

2. **Row reduces the matrix into reduced form.**

Row 2 – 3 ×row 1: $\Rightarrow \begin{bmatrix} 1 & -2 \mid -10 \\ 0 & 7 \mid \ 42 \end{bmatrix}$

$\frac{1}{7}$ ×row 2: $\Rightarrow \begin{bmatrix} 1 & -2 \mid -10 \\ 0 & 1 \mid \ 6 \end{bmatrix}$

Row 1+2×row 2: $\Rightarrow \begin{bmatrix} 1 & 0 \mid \ 2 \\ 0 & 1 \mid \ 6 \end{bmatrix}$

3. **Reform the equations from the reduced form.**

$x = 2$

$y = 6$

Solve by back substitution.

Because we have the matrix in reduced echelon form there is no need for back substitution; we just read off the solution from the reformed equations. Therefore the solution is x = 2, y = 6.

5. **Check answer.**

Substitute x = 2, y = 6 into (1)

$2 - 2\times 6 = -10$

$3\times 2 + 6 = 12$

Which are both true.

Example 2: Find the solution of the simultaneous equations

x + y = 5

4x + 2y = 10

−x + 5y = 14

1. **Create the augmented matrix.**

$$\begin{bmatrix} 1 & 1 \mid 5 \\ 4 & 2 \mid 10 \\ -1 & 5 \mid 14 \end{bmatrix}$$

2. **Row reduces the matrix into rank form.**

Row 2 – 4 ×row 1:

Row 3 + row 1: $\Rightarrow \begin{bmatrix} 1 & 1 \mid 5 \\ 0 & -2 \mid -10 \\ 0 & 6 \mid 19 \end{bmatrix}$

Row 3+3×row 2: $\Rightarrow \begin{bmatrix} 1 & 1 \mid 5 \\ 0 & -2 \mid -10 \\ 0 & 0 \mid -11 \end{bmatrix}$

3. **Reform the equations from the echelon form.**
x + y = 5
−2y = −10
0 = −11
In this case it is clear that the 3rd equation can never be true. This means that there are no values of x, y and z that simultaneously satisfy (3).

Linear system:

Rule 2: Consider the system of linear equations
$a_{11}x_1 + b_{11}y_1 = k_1$
$a_{21}x_1 + b_{21}x_2 = k_2$
Associated augmented matrix:

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \end{array}\right]$$

Operations that Produce Row-Equivalent Matrices:
Two rows are interchanged:
$R_i \leftrightarrow R_j$
A row is multiplied by a nonzero constant:
$kR_i \rightarrow R_i$
3. A constant multiple of one
row is added to another row:
$kR_j + R_i \rightarrow R_i$

Example 3: Solve using Augmented matrix:
Solve
x +3y=5
2x – y=3
1. Augmented system
2. Eliminate 2 in 2nd row by row operation
3. Divide row two by -7 to obtain a coefficient of 1.
4. Eliminate the 3 in first row, second position.
5. Read solution from matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right]$$

$$-2R_1 + R_2 \rightarrow \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -7 & -7 \end{array}\right]$$

$$R_2 / -7 \rightarrow R_2 \rightarrow \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

$$-3R_2 + R_1 \rightarrow R_1 \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \rightarrow x = 2, y = 1; (2,1)$$

Example 4: Solve a system using augmented matrix methods
x+2y=4
$x+\frac{1}{2}y=4$
Eliminate fraction in second equation.
Write system as augmented matrix.
Multiply row 1 by -2 and add to row 2
Divide row 2 by -3
Multiply row 2 by -2 and add to row 1.

$$x+2y=4$$
$$x+\frac{1}{2}y=4 \rightarrow 2x+y=8$$
$$\left[\begin{array}{cc|c}1 & 2 & 4\\2 & 1 & 8\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1 & 2 & 4\\0 & -3 & 0\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1 & 2 & 4\\0 & 1 & 0\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1 & 0 & 4\\0 & 1 & 0\end{array}\right]$$

Read solution : x = 4, y = 0
(4,0)

Example 5: Solve a system using augmented matrix methods
10x -2y=6
-5x+y= -3
1. Represent as augmented matrix.
2. Divide row 1 by 2
3. Add row 1 to row 2 and replace row 2 by sum
4. Since 0 = 0 is always true, we have a dependent system. The two equations are identical and there are an infinite number of solutions.

$$\left[\begin{array}{cc|c}10 & -2 & 6\\-5 & 1 & -3\end{array}\right] \rightarrow \left[\begin{array}{cc|c}5 & -1 & 3\\-5 & 1 & -3\end{array}\right] \rightarrow \left[\begin{array}{cc|c}5 & -1 & 3\\0 & 0 & 0\end{array}\right]$$

Example 6: Solve:

$$5x-2y=-7$$
$$y=\frac{5}{2}x+1$$

Rewrite second equation :

$$2y=5x+2$$
$$-5x+2y=2$$

Since we have an impossible equation, there is no solution. The two lines are parallel and do not intersect.

$$\left.\begin{array}{r}5x-2y=-7\\-5x+2y=2\end{array}\right\} \rightarrow \left[\begin{array}{cc|c}5 & -2 & -7\\-5 & 2 & 2\end{array}\right] \rightarrow \left[\begin{array}{cc|c}5 & -2 & -7\\0 & 0 & -5\end{array}\right]$$

A system of linear equations is transformed into an equivalent system if:

1) two equations are interchanged
2) an equation is multiplied by a nonzero constant
3) a constant multiple of one equation is added to another equation