

Sets

Definition 1: Sets: A set is a collection of objects, called elements of the set. A set can be represented by listing its elements between braces:

$A = \{1, 2, 3, 4, 5\}$. The symbol \in is used to express that an element is (or belongs to) a set, for instance $3 \in A$. Its negation is represented by \notin , e.g. $7 \notin A$. If the set is finite, its number of elements is represented $|A|$, e.g. if $A = \{1, 2, 3, 4, 5\}$ then $|A| = 5$.

Some important sets are the following:

1. $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ = the set of natural numbers.
2. $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ = the set of integers.
3. \mathbb{Q} = the set of rational numbers.
4. \mathbb{R} = the set of real numbers.
5. \mathbb{C} = the set of complex numbers.

Note 1: \mathbb{N} includes zero—for some authors $\mathbb{N} = \{1, 2, 3, \dots\}$, without zero.

Let S be one of those sets then we use the following notations:

1. S^+ = set of positive elements in S , for instance $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ = the set of positive integers.
2. S^- = set of negative elements in S , for instance $\mathbb{Z}^- = \{-1, -2, -3, \dots\}$ = the set of negative integers.
3. S^* = set of elements in S excluding zero, for instance \mathbb{R}^* = the set of non zero real numbers.

Definition 2: Set-builder notation: An alternative way to define a set, called set builder notation, is by stating a property (predicate) $P(x)$ verified by exactly its elements, for instance $A = \{x \in \mathbb{Z} \mid 1 \leq x \leq 5\}$ = “set of integers x such that $1 \leq x \leq 5$ ” i.e.: $A = \{1, 2, 3, 4, 5\}$. In general: $A = \{x \in U \mid p(x)\}$, where U is the universe of discourse in which the predicate $P(x)$ must be interpreted, or $A = \{x \mid P(x)\}$ if the universe of discourse for $P(x)$ is implicitly understood. In set theory the term universal set is often used in place of “universe of discourse” for a given predicate.

Definition 3: Principle of Extension: Two sets are equal if and only if they have the same elements, i.e.:

$$A = B \equiv \forall x (x \in A \leftrightarrow x \in B).$$

Subset: We say that A is a subset of set B , or A is contained in B , and we represent it “ $A \subseteq B$ ”, if all elements of A are in B ,

Example 1: if $A = \{a, b, c\}$ and $B = \{a, b, c, d, e\}$ then $A \subseteq B$.

Definition 4: A is a proper subset of B , represented “ $A \subset B$ ”, if $A \subseteq B$ but $A \neq B$, i.e., there is some element in B which is not in A .

Definition 5: Empty Set. A set with no elements is called empty set (or null set, or void set), and is represented by \emptyset or $\{\}$.

Note 2: Nothing prevents a set from possibly being an element of another set (which is not the same as being a subset!).

Example 2: if $A = \{1, a, \{3, t\}, \{1, 2, 3\}\}$ and $B = \{3, t\}$, then obviously B is an element of A , i.e., $B \in A$.

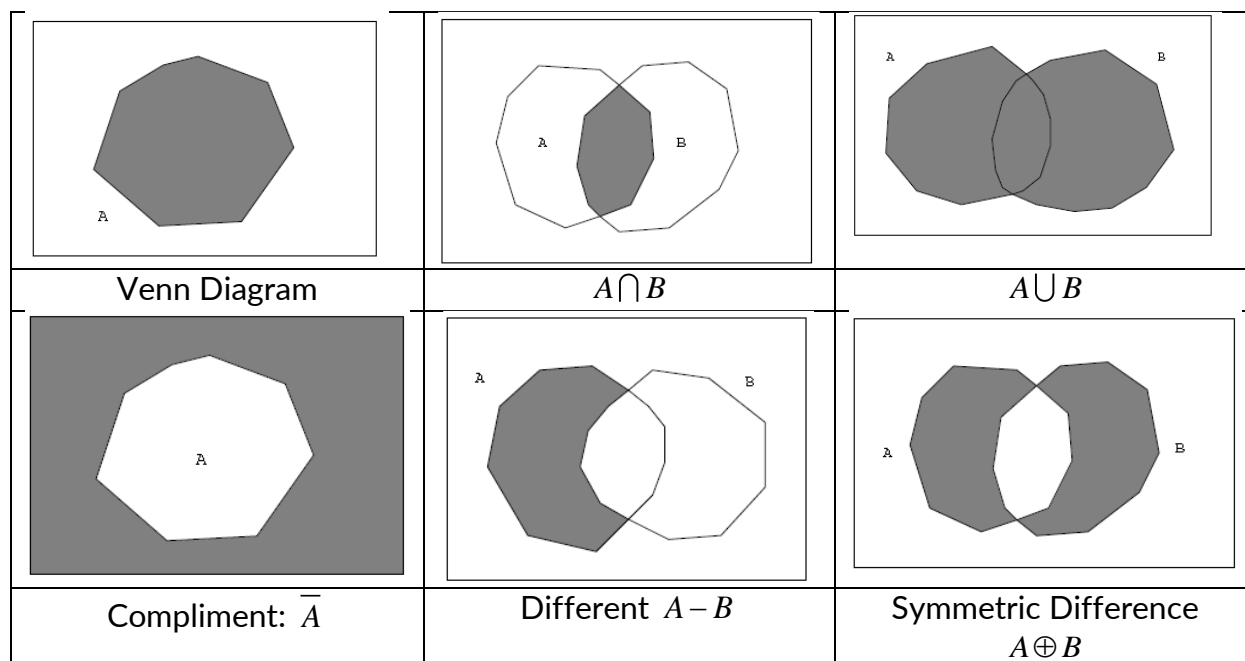
Definition 6: Power Set: The collection of all subsets of a set A is called the power set of A , and is represented $P(A)$.

Example 3: if $A = \{1, 2, 3\}$, then $P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}$.

Definition 7: Multi-sets: Two ordinary sets are identical if they have the same elements, so for instance, $\{a, a, b\}$ and $\{a, b\}$ are the same set because they have exactly the same elements, namely a and b . However, in some applications it might be useful to allow repeated elements in a set. In that case we use multi-sets, which are mathematical entities similar to sets, but with possibly repeated elements. So, as multi-sets, $\{a, a, b\}$ and $\{a, b\}$ would be considered different, since in the first one the element a occurs twice and in the second one it occurs only once.

Venn Diagrams

Venn diagrams are graphic representations of sets as enclosed areas in the plane. For instance, in the figure, the rectangle represents the universal set (the set of all elements considered in a given problem) and the shaded region represents a set A . The other figures represent various set operations.



Set Operations

Intersection: The common elements of two sets

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

If $A \cap B = \emptyset$, the sets are said to be disjoint.

Union: The set of elements that belong to either of two sets

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

Complement: The set of elements (in the universal set) that do not belong to a given set:

$$\overline{A} = \{x \in U \mid x \notin A\}$$

Difference or relative Complement: The set of elements that belongs to a set but not to another:

$$A - B = \{x \mid (x \in A) \wedge (x \notin B)\} = A \cap \overline{B}$$

Symmetric Difference: Given two sets, their symmetric difference is the set of elements that belongs to either one or the other set but not both.

$$A \oplus B = \{x \mid (x \in A) \oplus (x \in B)\}$$

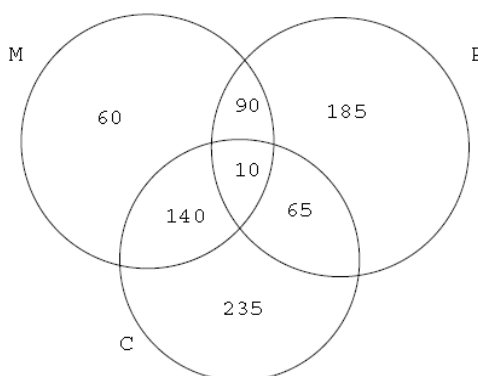
It can be expressed also in the following way:

$$A \oplus B = A \cup B - A \cap B = (A - B) \cup (B - A)$$

Counting with Venn Diagrams:

A Venn diagram with n sets intersecting in the most general way divides the plane into 2^n regions. If we have information about the number of elements of some portions of the diagram, then we can find the number of elements in each of the regions and use that information for obtaining the number of elements in other portions of the plane.

Example 4: Let M , P and C be the sets of students taking Mathematics courses, Physics courses and Computer Science courses respectively in a university. Assume $|M| = 300$, $|P| = 350$, $|C| = 450$, $|M \cap P| = 100$, $|M \cap C| = 150$, $|P \cap C| = 75$, $|M \cap P \cap C| = 10$. How many students are taking exactly one of those courses?



We see that $(M \cap P) - (M \cap P \cap C) = 100 - 10 = 90$, $(M \cap C) - (M \cap P \cap C) = 150 - 10 = 140$ and $(P \cap C) - (M \cap P \cap C) = 75 - 10 = 65$.

Then the region corresponding to students taking Mathematics courses only has cardinality $300 - (90 + 10 + 140) = 60$. Analogously we compute the number of students taking Physics courses only (185) and taking Computer Science courses only (235).

The sum $60 + 185 + 235 = 480$ is the number of students taking exactly one of those courses.