Rational Functions

The quotient of two algebraic expressions is a fractional expression. More, over, the quotient of two polynomials such as $\frac{1}{x}, \frac{2x+1}{x-1}, \frac{x^2-4}{x^2+9}$ is a rational expression.

Definition 1: A rational function is basically a division of two polynomial functions. That is, it is a polynomial divided by another polynomial. In formal notation, a rational function would be symbolized like this:

$$f(x)=\frac{s(x)}{t(x)}$$

Where $s(x)$ and $t(x)$ are polynomial functions, and $t(x)\neq 0$.

Since division by zero is undefined, we need to be careful when substituting values for the variable in a rational expression.

Definition 2: **Domain of a Rational Function:** The domain of a rational expression is the set of all values for the variable that make the expression defined.

The domain of the rational function $f(x)=\frac{s(x)}{t(x)}$ consists of all points x where $t(x)\neq 0$.

Example 1: Find the domain of $\frac{3x-1}{x-8}$

The polynomial in the denominator is $x-8$. So, $x-8$ must be different fom zero.

$x-8\neq 0 \Rightarrow x\neq 8$

Therefore, the domain of $\frac{3x-1}{x-8}$ is $x\neq 8$. In interval notation $x\in(-\infty,8)\cup(8,\infty)$

Example 2: Find all numbers that must be excluded from the domain of:

$$\frac{x+5}{x^2+3x-4}$$

To find the excluded values, factorize the denominator then find the values that will make the denominator equal to zero.

$$\frac{x+5}{x^2+3x-4}=\frac{x+5}{(x+4)(x-1)}$$

$(x+4)(x-1)\neq 0 \Rightarrow x\neq -4 \,\&\, x\neq 1$

So, the excluded values are -4 and 1

Recall that a fraction is written in lowest terms if the numerator and denominator have no common factors other that 1 or -1. Simplifying a rational expression follows the same idea. To write a rational expression in lowest form we use the following principle.

Fundamental Principle of Rational Expressions

For any rational expression $\frac{s(x)}{t(x)}$, and any polynomial $R(x)$, where, $R(x) \neq 0$, then

$$\frac{s(x)R(x)}{t(x)R(x)} = \frac{s(x)}{t(x)}$$

In other words, if you multiply or divide by the EXACT SAME polynomial the numerator and denominator, then you have an equivalent rational expression.
This will come in handy when we simplify rational expressions, which is coming up next.

Simplifying (or reducing) a Rational Expression

Step 1: Factor the numerator and the denominator completely.
Step 2: Mention all the excluded values, find the domain of definition.
Step 3: Apply the fundamental principle of rational expressions to divide out all common factors that the numerator and the denominator have.

Sometimes, the process of factorizing will be very important in simplifying fractions. Here are some examples of possible simplifications, and some warnings of what *can't* be done.
If you have always found this sort of thing difficult, it may help you here to highlight the matching parts which are cancelling with each other in the same color.

Example 3: Simplify the rational function if possible and state all the excluded values:

1) $\frac{xy+xz}{xw} = \frac{\cancel{x}(y+z)}{\cancel{x}w} = \frac{y+z}{w}$ Dividing top and bottom by x $x \neq 0$, $w \neq 0$

2) $\frac{ab+ac}{b+c} = \frac{a\cancel{(b+c)}}{\cancel{(b+c)}} = \frac{a}{1} = a$ Dividing top and bottom by $(b+c)$ $b \neq -c$

3) $\frac{ab+c}{b+c}$ can not be simplified. We can not cancel $(b+c)$ $b \neq -c$

4) $\frac{x+xy}{x^2} = \frac{\cancel{x}(1+y)}{\cancel{x^2}} = \frac{1+y}{x}$ Divide top and bottom by x $x \neq 0$

5) $\frac{x^2+5x+6}{x^2-3x-10} = \frac{(x+3)\cancel{(x+2)}}{(x-5)\cancel{(x+2)}} = \frac{x+3}{x-5}$ Divide top and bottom by $(x+2)$ $x \neq 5$, $x \neq -2$

Example 4: Find the domain of definition of each tate the non-permissible values in the rational function:

1) $f(x)=\frac{x+3}{2x^2-18x}$.

Set the denominator equal to zero. $2x^2-18x=0$

$2x(x-9)=0$

Set each factor equal to zero. $2x=0$ or $x-9=0$

$x=0$ and $x=9$

The excluded values for this rational function are $x=0,\ 9$.

The **domain** of this rational function is all real numbers except $x=0,\ 9$.

$$D=\{x \mid x \in \mathbb{R}, x\neq 0,9\}$$

2) $f(x)=\frac{4}{x-2}$.

Set the denominator equal to zero: $x-2=0 \quad \Rightarrow \quad x=2$

The excluded value for this rational function is $x=2$.

The domain of this rational function is all real numbers except $x=2$.

$$D=\{x \mid x\in \mathbb{R}, x\neq 2\}$$